ON MEASURE SOLUTIONS OF THE BOLTZMANN EQUATION, PART I: MOMENT PRODUCTION AND STABILITY ESTIMATES

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Abstract. The spatially homogeneous Boltzmann equation with hard potentials is considered for measure valued initial data having finite mass and energy. We prove the existence of *weak measure solutions*, with and without angular cutoff on the collision kernel; the proof in particular makes use of an approximation argument based on the *Mehler transform*. Moment production estimates in the usual form and in the exponential form are obtained for these solutions. Finally for the Grad angular cutoff, we also establish uniqueness and strong stability estimate on these solutions.

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Contents

1. Introduction

In this paper we study the spatially homogeneous Boltzmann equation for hard interaction potentials with or without angular cutoff. The initial data are assumed to be positive Borel measures having finite moments up to order 2. Our main results are the existence and stability of measure solutions that have polynomial and exponential moment production properties.

1.1. The spatially homogeneous Boltzmann equation.

1.1.1. *The equation.* Before introducing the main results, let us recall the Boltzmann equation for L^1-solutions and basic notations. The equation for the space homogeneous solution takes the form

$$\frac{\partial}{\partial t} f_t(v) = Q(f_t, f_t)(v)\,, \quad (v,t) \in \mathbb{R}^N \times (0,\infty)\,, \quad N \geq 2 \tag{1.1}$$

with some given initial data $f_t(v)|_{t=0} = f_0(v)$ and Q is the collision integral defined by

$$Q(f,f)(v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_*, \sigma)\Big(f(v')f(v'_*) - f(v)f(v_*)\Big) \mathrm{d}\sigma \mathrm{d}v_*\,, \tag{1.2}$$

where v, v_* and v', v'_* stand for velocities of two particles respectively after and before their collision,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma\,, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma\,, \quad \sigma \in \mathbb{S}^{N-1}\,. \tag{1.3}$$

The above relation between v, v_* and v', v'_* shows that the collision is elastic:

$$v' + v'_* = v + v_*\,, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2\,.$$

1.1.2. *The collision kernel.* The collision kernel $B(z, \sigma)$ under consideration is assumed to be a function of $(|z|, \frac{z}{|z|} \cdot \sigma)$, i.e.

$$B(z, \sigma) = \bar{B}(|z|, \cos\theta), \quad \cos\theta = \frac{z}{|z|} \cdot \sigma, \quad \theta \in [0, \pi] \tag{1.4}$$

where $(r,t) \mapsto \bar{B}(r,t)$ is a non-negative Borel function on $[0,\infty) \times [-1,1]$ satisfying

$$\forall\, t \in (-1,1), \quad r \mapsto \bar{B}(r,t) \text{ is continuous on } [0,\infty), \tag{1.5}$$

$$\bar{B}(r,t) \leq (1 + r^2)^{\gamma/2} b(t), \quad 0 < \gamma \leq 2. \tag{1.6}$$

In this paper most of the results are concerned with the case

$$B(z,\sigma) = |z|^\gamma b(\cos\theta), \quad 0 < \gamma \leq 2 \tag{1.7}$$

which corresponds to the so-called hard potential molecular interactions.

The function $t \mapsto b(t)$ in (1.6)-(1.7) has some weighted integrability. We shall consider several options for the assumptions on $b(\cdot)$. Our strongest assumption is that $b(\cdot)$ as a function of σ is integrable on the sphere $\mathbb{S}^{N-1}$, which means

$$\int_0^\pi b(\cos\theta) \sin^{N-2}\theta \,\mathrm{d}\theta < \infty$$

which is the Grad's angular cutoff. However more singular situations can be considered. The minimal assumption is that $b(\cos\theta)\sin^2\theta$ is integrable on the sphere as a function of σ (this corresponds physically to an angular momentum), i.e.

$$\int_0^\pi b(\cos\theta) \sin^N\theta \,\mathrm{d}\theta < \infty\,.$$

In dimension $N = 3$, it is well known that for the hard spheres model the function $b(\cdot)$ is constant, whereas for hard potential models (without angular cutoff), there is only weighted integrability:

$$\int_0^\pi b(\cos\theta)\sin\theta \,\mathrm{d}\theta = \infty, \quad \int_0^\pi b(\cos\theta)\sin^2\theta \,\mathrm{d}\theta < \infty\,.$$

More precisely, given an interaction potential$\phi(r) = C\, r^{1-s}$ for $C > 0$ and $s > 3$, we obtain the following formula from the physics literature [11] in dimension $N = 3$:

$$B(z,\sigma) = |z|^{\gamma} b(\cos\theta), \qquad \gamma = \frac{s-5}{s-1}\,;$$
$$b(\cos\theta)\sin\theta \sim C'\,\theta^{-1-\frac{2}{s-1}} \quad (\theta \to 0^+)$$

for some constant $C' > 0$, and hard potential interactions correspond to $s > 5$.

In this paper we consider the following different assumptions:

$$(\mathbf{H0}) \qquad 0 < \gamma \le 2\,, \quad A_2 := \left|\mathbb{S}^{N-2}\right| \int_0^{\pi} b(\cos\theta)\sin^N\theta\,\mathrm{d}\theta < \infty,$$

$$(\mathbf{H1}) \qquad 0 < \gamma \le 2\,, \quad \int_0^{\pi} b(\cos\theta)\sin^N\theta\,(1 + |\log(\sin\theta)|)\mathrm{d}\theta < \infty,$$

$$(\mathbf{H2}) \qquad 1 < \gamma < 2\,, \quad \int_0^{\pi} b(\cos\theta)\sin^{N-2\nu}\theta\,\mathrm{d}\theta < \infty\,, \quad \nu = 2 - 2/\gamma \in (0,1),$$

$$(\mathbf{H3}) \qquad \gamma = 2\,,\ \exists\, p \in (1,\infty) \quad \text{s.t.} \quad \int_0^{\pi} [b(\cos\theta)]^p \sin^{N-2}\theta\,\mathrm{d}\theta < \infty,$$

$$(\mathbf{H4}) \qquad 0 < \gamma \le 2\,, \quad A_0 := \left|\mathbb{S}^{N-2}\right| \int_0^{\pi} b(\cos\theta)\sin^{N-2}\theta\,\mathrm{d}\theta < \infty.$$

Observe that $(\mathbf{H3})|_b \Rightarrow (\mathbf{H4})|_b \Rightarrow (\mathbf{H2})|_b \Rightarrow (\mathbf{H1})|_b \Rightarrow (\mathbf{H0})|_b$, where for instance $(\mathbf{H3})|_b$ denotes the assumption with respect to $b(\cdot)$ in $(\mathbf{H3})$. Note also that $(\mathbf{H3})|_b$ and $(\mathbf{H4})|_b$ corresponds to the angular cutoff case (short-range interactions), whereas $(\mathbf{H0})|_b, (\mathbf{H1})|_b$ and $(\mathbf{H2})|_b$ allow for non-locally integrable functions $b(\cdot)$ on the sphere, i.e. non-cutoff cases (long-range interactions).

1.1.3. *Dual form of the collision operator.* For any $\mathbf{n} \in \mathbb{S}^{N-1}$, let

$$\mathbb{S}^{N-2}(\mathbf{n}) = \{\omega \in \mathbb{S}^{N-1} \mid \omega\cdot\mathbf{n} = 0\,\} \quad (N \ge 3)$$

and in dimension $N = 2$ let

$$\mathbb{S}^0(\mathbf{n}) = \{-\mathbf{n}^{\perp}\,,\ \mathbf{n}^{\perp}\} \quad \text{where } \mathbf{n}^{\perp} \in \mathbb{S}^1 \text{ satisfies } \mathbf{n}^{\perp}\cdot\mathbf{n} = 0\,.$$

Then for any $g \in L^1(\mathbb{S}^{N-1})$ or $g \ge 0$ (measurable) on $\mathbb{S}^{N-1}$ we have

$$\int_{\mathbb{S}^{N-1}} g(\sigma)\mathrm{d}\sigma = \int_0^{\pi} \sin^{N-2}\theta \left(\int_{\mathbb{S}^{N-2}(\mathbf{n})} g(\cos\theta\mathbf{n} + \sin\theta\,\omega)\mathrm{d}\omega\right)\mathrm{d}\theta$$

where $\mathrm{d}\omega$ is the Lebesgue spherical measure on $\mathbb{S}^{N-2}(\mathbf{n})$ and in case $N = 2$ we define

$$\int_{\mathbb{S}^0(\mathbf{n})} g(\omega)\mathrm{d}\omega = g(-\mathbf{n}^{\perp}) + g(\mathbf{n}^{\perp})\,.$$

Let $|\mathbb{S}^{N-2}(\mathbf{n})| = \int_{\mathbb{S}^{N-2}(\mathbf{n})}\mathrm{d}\omega$, etc. Then $|\mathbb{S}^{N-2}(\mathbf{n})| = |\mathbb{S}^{N-2}|$ for $N \ge 3$, $|\mathbb{S}^0(\mathbf{n})| = |\mathbb{S}^0| = 2$ for $N = 2$.

By classical calculation one has

$$\langle Q(f,g),\varphi\rangle := \int_{\mathbb{R}^N} Q(f,g)(v)\varphi(v)\mathrm{d}v = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B[\Delta\varphi](v,v_*)f(v)g(v_*)\mathrm{d}v\mathrm{d}v_* \tag{1.8}$$

where

$$\Delta\varphi := \Delta\varphi(v,v_*,v',v'_*) = \varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*)\,,$$

$$L_B[\Delta\varphi](v,v_*) := \int_0^\pi \bar{B}(|v-v_*|,\cos\theta)\sin^{N-2}\theta\left(\int_{\mathbb{S}^{N-2}(\mathbf{n})}\Delta\varphi\,\mathrm{d}\omega\right)\mathrm{d}\theta \tag{1.9}$$

and $\sigma = \cos\theta\,\mathbf{n}+\sin\theta\,\omega$, $\mathbf{n} = (v-v_*)/|v-v_*|$ for $v \neq v_*$; $\mathbf{n} = \mathbf{e}_1 = (1,0,\dots,0)$ for $v = v_*$.

Observe that when assuming one of the assumptions **(H0)**, **(H1)**, **(H2)** (non-cutoff cases), the collision operator in the dual form (1.8) above is well-defined thanks to the cancellations in the symmetric difference $\Delta\varphi$ of $\varphi \in C^2(\mathbb{R}^N)$. Basic estimates on $\Delta\varphi$ are as follows (see for instance [10, Lemma 3.2]): For all $(v,v_*,\sigma) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}$ one has

$$|\Delta\varphi| \le \sqrt{2}\left(\max_{|\xi|\le\sqrt{|v|^2+|v_*|^2}}|\nabla\varphi(\xi)|\right)|v-v_*|\sin\theta\,; \tag{1.10}$$

$$\left|\int_{\mathbb{S}^{N-2}(\mathbf{n})}\Delta\varphi\,\mathrm{d}\omega\right| \le |\mathbb{S}^{N-2}|\left(\max_{|\xi|\le\sqrt{|v|^2+|v_*|^2}}|H_\varphi(\xi)|\right)|v-v_*|^2\sin^2\theta\,, \tag{1.11}$$

where $\nabla\varphi$, H_φ are gradient and Hessian matrix of φ. Consequently the Boltzmann equation (1.1) in a weak form can be written

$$\int_{\mathbb{R}^N}\varphi(v)f_t(v)\mathrm{d}v = \int_{\mathbb{R}^N}\varphi(v)f_0(v)\mathrm{d}v + \int_0^t\langle Q(f_\tau,f_\tau),\varphi\rangle\mathrm{d}\tau\,. \tag{1.12}$$

From the estimate (1.11) it is easily seen that if $A_2 < \infty$ (minimal assumption) then $L_B[\Delta\varphi]$ is well-defined for all $\varphi \in C^2(\mathbb{R}^N)$.

In fact we shall prove in Proposition 2.1 (see Section 2) that $(v,v_*)\mapsto L_B[\Delta\varphi](v,v_*)$ is also continuous on $\mathbb{R}^N\times\mathbb{R}^N$. Furthermore if

$$\int_0^\pi b(\cos\theta)\sin^{N-1}\theta\,\mathrm{d}\theta < \infty$$

then from the estimate (1.10) one sees that

$$L_B[|\Delta\varphi|](v,v_*) = \int_{\mathbb{S}^{N-1}} B(v-v_*,\sigma)|\Delta\varphi|\,\mathrm{d}\sigma < \infty$$

so that L_B coincides with the simpler formula

$$L_B[\Delta\varphi](v,v_*) = \int_{\mathbb{S}^{N-1}} B(v-v_*,\sigma)\Delta\varphi\,\mathrm{d}\sigma\,. \tag{1.13}$$

The collision integral (1.8) and the equation (1.12) for L^1-functions are naturally extended to finite Borel measures. For every $0\le s<\infty$, let $\mathcal{B}_s(\mathbb{R}^N) = (\mathcal{B}_s(\mathbb{R}^N),\|\cdot\|_s)$ be the Banach space of real Borel measures on $\mathbb{R}^N$ having finite total variations up to order s, i.e.

$$\|\mu\|_s := \int_{\mathbb{R}^N}\langle v\rangle^s\mathrm{d}|\mu|(v) < \infty, \quad \langle v\rangle := (1+|v|^2)^{1/2}$$

where the positive Borel measure $|\mu|$ is the total variation of μ. In particular $\|\mu\| = \|\mu\|_0 = |\mu|(\mathbb{R}^N)$ is simply the total variation of μ. Let

$$\mathcal{B}_s^+(\mathbb{R}^N) = \left\{\mu\in\mathcal{B}_s(\mathbb{R}^N)\mid \mu\ge 0\right\}\,.$$

In accordance with (1.8) we now define for every $\mu,\nu\in\mathcal{B}_s(\mathbb{R}^N)$ and every suitable smooth function φ

$$\langle Q(\mu,\nu),\,\varphi\rangle := \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B[\Delta\varphi](v,v_*)\mathrm{d}\mu(v)\mathrm{d}\nu(v_*)\,. \tag{1.14}$$

Our test function space for defining measure weak solutions is chosen $C_b^2(\mathbb{R}^N)$, where

$$C_b^k(\mathbb{R}^N) = \left\{ \varphi \in C^k(\mathbb{R}^N) \,\Big|\, \sum_{|\alpha| \le k} \sup_{v \in \mathbb{R}^N} |\partial^\alpha \varphi(v)| < \infty \right\}.$$

Finally by analogy with $\mathcal{B}_s(\mathbb{R}^N)$ we introduce the class $L^\infty_{-s}(\mathbb{R}^N)$ of *locally bounded* Borel functions such that

$$\psi \in L^\infty_{-s}(\mathbb{R}^N) \iff \|\psi\|_{L^\infty_{-s}} := \sup_{v \in \mathbb{R}^N} |\psi(v)| \langle v \rangle^{-s} < \infty$$

and we define

$$L^\infty_{-s} \cap C^k(\mathbb{R}^N) = \left\{ \varphi \in C^k(\mathbb{R}^N) \,\Big|\, \sum_{|\alpha| \le k} \|\partial^\alpha \varphi\|_{L^\infty_{-s}} < \infty \right\}, \quad s \ge 0,\ k \in \mathbb{N}.$$

1.2. **Previous results and references.** Let us give a short (and non-exhaustive) overview of the main previous results and references related to the subject of this paper.

1.2.1. *Cauchy theory for the spatially homogeneous Boltzmann equation for hard potentials with cutoff.* The first rigorous mathematical result is due to Carleman [8, 9] who proved existence and uniqueness of solutions in $L^1 \cap L^\infty$ with pointwise moment bounds, for hard spheres interactions. A general Cauchy theory was later developed by Arkeryd [4, 5] who proved existence and uniqueness of solutions in $L^1 \cap L \log L$ with L^1 moment bounds. More recently optimal results were obtained by Mischler and Wennberg [23] (see also Lu [20]), and we refer to the references therein for a more extensive bibliography.

1.2.2. *Cauchy theory for the spatially homogeneous Boltzmann equation for hard potentials without cutoff.* This theory is much more recent, and not complete at now. As far as existence of solutions is concerned let us mention the seminal works of Villani [28] and then Alexandre and Villani [2]. As far as uniqueness of solutions is concerned (in the general far from equilibrium regime), let us mention the works [27, 15, 17, 16] based on Wasserstein metrics and probabilistic tools, and the work [13] based on *a priori* estimates. Finally let us mention the related recent works in the *perturbative close-to-equilibrium regime* (but without assuming spatial homogeneity) of Gressman and Strain [19] on the one hand, and Alexandre, Morimoto, Ukai, Xu, Yang [1] on other hand.

1.2.3. *Polynomial moment bounds.* The first seminal result of the propagation of polynomial moments that exists initially for "variable hard spheres" (hard potentials with angular cutoff) is due to Elmroth [14] and makes use of so-called "Povzner's inequalities" [25]. Then Desvillettes [12] proved, for the same model, the appearance of any polynomial as soon as a moment of order strictly higher than 2 exists initially (see also [29]). Finally optimal results were obtained in [23] again.

1.2.4. *Exponential moment bounds.* The first seminal result of propagation of moments of exponential form is due to Bobylev [6], still in the case of short-ranged interactions. Significant improvements of these results were later obtained in [7]. Let us also mention the related result of propagation of pointwise Maxwellian bound in [18]. Inspired by the same techniques, the appearance of exponential moments was first obtained by the second author together with Mischler in [22, 24], see also the recent work [3].

1.3. **Definitions of measure solutions.** Let us start with a notion of *measure weak solutions*, where the time evolution is defined in the integral sense.

Definition 1.1 (Measure weak solutions)**.** Let $B(z,\sigma)$ be given by (1.4)-(1.5)-(1.6) with γ and $b(\cdot)$ satisfying **(H0)**. Let $F_0 \in \mathcal{B}_2^+(\mathbb{R}^N)$ and $\{F_t\}_{t\ge 0} \subset \mathcal{B}_2^+(\mathbb{R}^N)$. We say that $\{F_t\}_{t\ge0}$, or simply F_t, is a **measure weak solution** of Eq. (1.1) associated with the initial datum F_0, if it satisfies the following (i)-(ii):

(i) $\sup_{t\ge 0} \|F_t\|_2 < \infty$.

(ii) For every $\varphi \in C_b^2(\mathbb{R}^N)$,

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^N\times\mathbb{R}^N} |L_B[\Delta\varphi](v,v_*)|\mathrm{d}F_t(v)\mathrm{d}F_t(v_*) < \infty\,, \quad \forall\, t>0 \\ t \mapsto \langle Q(F_t,F_t),\,\varphi\rangle \quad \text{belongs to} \quad C((0,\infty))\cap L^1_{\rm loc}([0,\infty)) \\ \displaystyle\int_{\mathbb{R}^N} \varphi(v)\mathrm{d}F_t(v) = \int_{\mathbb{R}^N}\varphi(v)\mathrm{d}F_0(v) + \int_0^t \langle Q(F_\tau,F_\tau),\,\varphi\rangle\mathrm{d}\tau \quad \forall\, t\ge 0\,. \end{cases}$$

Moreover a measure weak solution F_t is called a **conservative solution** if it conserves the mass, momentum and energy, i.e.

$$\int_{\mathbb{R}^N}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}F_t(v) = \int_{\mathbb{R}^N}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}F_0(v) \qquad \forall\, t\ge 0\,.$$

Note that every measure weak solution conserves the mass because the constant $\varphi = 1$ belongs to $C_b^2(\mathbb{R}^N)$ and $\Delta\varphi = 0$. The conservations of the momentum and energy are formally true since one also has $\Delta\varphi = 0$ for $\varphi = v_j$, $j = 1,2,\dots,N$ and $\varphi = |v|^2$, but these φ do not belong to $C_b^2(\mathbb{R}^N)$. In fact under the assumption **(H1)**, one can follow the same argument in [21] to construct a weak solution of Eq. (1.1) such that the energy is increasing.

Now let us consider a stronger notion of *measure strong solutions* under the angular cutoff assumption **(H4)**. Let $B(z,\sigma)$ be given by (1.4)-(1.5)-(1.6) with $b(\cdot)$ satisfying $A_0 < \infty$. Then we can define bilinear operators (see Proposition 2.3 below)

$$Q^\pm : \mathcal{B}_{s+\gamma}(\mathbb{R}^N)\times\mathcal{B}_{s+\gamma}(\mathbb{R}^N)\to\mathcal{B}_s(\mathbb{R}^N) \quad (s\ge 0)$$

and

$$Q(\mu,\nu) := Q^+(\mu,\nu) - Q^-(\mu,\nu) \tag{1.15}$$

through Riesz's representation theorem by

$$\int_{\mathbb{R}^N}\psi(v)\mathrm{d}Q^+(\mu,\nu)(v) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B[\psi](v,v_*)\mathrm{d}\mu(v)\mathrm{d}\nu(v_*)\,, \tag{1.16}$$

$$\int_{\mathbb{R}^N}\psi(v)\mathrm{d}Q^-(\mu,\nu)(v) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\psi(v)d\mu(v)\mathrm{d}\nu(v_*) \tag{1.17}$$

for all $\psi \in L^\infty_{-s}\cap C(\mathbb{R}^N)$, where

$$L_B[\psi](v,v_*) = \int_{\mathbb{S}^{N-1}} B(v-v_*,\sigma)\psi(v')\,\mathrm{d}\sigma, \quad A(z) = \int_{\mathbb{S}^{N-1}} B(z,\sigma)\mathrm{d}\sigma \tag{1.18}$$

and recall that $\mathbf{n} = (v - v_*)/|v - v_*|$ in $b(\mathbf{n} \cdot \sigma)$ is replaced by a fixed unit vector $\mathbf{e}_1$ for $v = v_*$.

Recall that the norm $\|\mu\|_s$ of $\mu \in \mathcal{B}_s(\mathbb{R}^N)$ $(s \geq 0)$ can be estimated in terms of compactly smooth test functions: For all $k \geq 0$

$$\|\mu\|_s = \sup_{\varphi \in C_c^k(\mathbb{R}^N),\, \|\varphi\|_{L^\infty_{-s}} \leq 1} \left| \int_{\mathbb{R}^N} \varphi \mathrm{d}\mu \right| . \tag{1.19}$$

We are now ready for stating the definition of *measure strong solutions*, for which some time-differentiability is assumed in total variation topology.

Definition 1.2 (Measure strong solutions)**.** Let $B(z, \sigma)$ be given by (1.4)-(1.5)-(1.6) with γ and $b(\cdot)$ satisfying **(H4)**. Let $F_0 \in \mathcal{B}_2^+(\mathbb{R}^N)$ and $\{F_t\}_{t \geq 0} \subset \mathcal{B}_2^+(\mathbb{R}^N)$. We say that F_t is a **measure strong solution** of Eq.(1.1) associated with the initial datum $F_t|_{t=0} = F_0$, if it satisfies the following (i)-(ii):

(i) $\sup_{t \geq 0} \|F_t\|_2 < \infty$.

(ii) $t \mapsto F_t \in C([0, \infty); \mathcal{B}_2(\mathbb{R}^N)) \cap C^1([0, \infty); \mathcal{B}_0(\mathbb{R}^N))$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = Q(F_t, F_t) , \quad t \in [0, \infty) . \tag{1.20}$$

Note that from (2.18)-(2.19)-(2.20) in Proposition 2.3 the strong continuity of

$$t \mapsto F_t \in C([0, \infty); \mathcal{B}_2(\mathbb{R}^N))$$

implies the strong continuity $t \mapsto Q(F_t, F_t) \in C([0, \infty); \mathcal{B}_0(\mathbb{R}^N))$, so that the differential equation (1.20) is equivalent to the integral equation

$$F_t = F_0 + \int_0^t Q(F_s, F_s) \mathrm{d}s , \quad t \geq 0 \tag{1.21}$$

where the integral is taken in the Riemann sense or generally in the Bochner sense. Recall also that here the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t$ and integral $\int_a^b \nu_t \mathrm{d}t$ as measures are defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \mu_t \right)(E) = \frac{\mathrm{d}}{\mathrm{d}t} \mu_t(E) , \quad \left(\int_a^b \nu_t \mathrm{d}t \right)(E) = \int_a^b \nu_t(E) \mathrm{d}t$$

for all Borel sets $E \subset \mathbb{R}^N$.

Note also that if a strong measure solution F_t is absolutely continuous with respect to the Lebesgue measure for all $t \geq 0$, i.e. $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v$, then it is easily seen that f_t (after modification on a v-null set) is a mild solution of Eq. (1.1). That is, $(t, v) \mapsto f_t(v)$ is nonnegative and Lebesgue measurable on $[0, \infty) \times \mathbb{R}^N$ and for every $t \geq 0$, $v \mapsto f_t(v)$ belongs to $L^1_2(\mathbb{R}^N)$, $\sup_{t \geq 0} \|f_t\|_{L^1_2} < \infty$, and there is a Lebesgue null set $Z_0 \subset \mathbb{R}^N$ (which is independent of t) such that

$$\begin{cases} \displaystyle\int_0^t Q^{\pm}(f_\tau, f_\tau)(v) \mathrm{d}\tau < \infty \quad \forall\, t \in [0, \infty) , \quad \forall\, v \in \mathbb{R}^N \setminus Z_0 \\ \displaystyle f_t(v) = f_0(v) + \int_0^t Q(f_\tau, f_\tau)(v) \mathrm{d}\tau , \quad \forall\, t \in [0, \infty) , \quad \forall\, v \in \mathbb{R}^N \setminus Z_0 . \end{cases}$$

Here

$$L^1_s(\mathbb{R}^N) = \left\{ f \in L^1(\mathbb{R}^N) \,\middle|\, \|f\|_{L^1_s} := \int_{\mathbb{R}^N} |f(v)| \langle v \rangle^s \mathrm{d}v < \infty \right\} , \quad s \geq 0 .$$

From classical measure theory [26, Theorem 6.13, page 149]: if $\mathrm{d}\mu(v) = f(v)\mathrm{d}v$ for $f \in L^1_s(\mathbb{R}^N)$, then $\mathrm{d}|\mu|(v) = |f(v)|\mathrm{d}v$ and hence $\|\mu\|_s = \|f\|_{L^1_s}$.

For any positive measure $\mu \in \mathcal{B}^+_2(\mathbb{R}^N)$ we finally introduce the following continuous function $r \mapsto \Psi_\mu(r)$ on $[0,\infty)$:

$$\Psi_\mu(r) = r + r^{1/3} + \int_{|v|>r^{-1/3}} |v|^2 \mathrm{d}\mu(v)\,, \quad r > 0 \quad \text{with} \quad \Psi_\mu(0) = 0 \tag{1.22}$$

which quantifies the localization of the energy of μ.

1.4. **Main results.** Our first main result is the following

Theorem 1.3 (Existence of solutions and moment production estimates without cutoff)**.** *Suppose that $B(z,\sigma) = |z|^\gamma b(\cos\theta)$ satisfies* **(H1)**. *Given any initial datum $F_0 \in \mathcal{B}^+_2(\mathbb{R}^N)$ with $\|F_0\|_0 \neq 0$, we have*

(a) *The Eq.* (1.1) *always has a conservative measure weak solution F_t satisfying $F_t|_{t=0} = F_0$.*

(b) *Let F_t be a measure weak solution of Eq.* (1.1) *associated with the initial datum F_0 satisfying*

$$\|F_t\|_2 \le \|F_0\|_2 \quad \forall\, t > 0; \quad \sup_{t\ge t_0} \|F_t\|_s < \infty \quad \forall\, t_0 > 0,\ \forall\, s > 2. \tag{1.23}$$

Then F_t is conservative, i.e. F_t conserves the mass, momentum, and energy.

(c) *The Eq.* (1.1) *always has a conservative measure weak solution F_t with $F_t|_{t=0} = F_0$ which satisfies the following moment production estimate:*

$$\|F_t\|_s \le \mathcal{K}_s(F_0)\left(1 + \frac{1}{t}\right)^{\frac{s-2}{\gamma}} \quad \forall\, t > 0\,, \quad \forall\, s \ge 2 \tag{1.24}$$

where

$$\mathcal{K}_s(F_0) = \|F_0\|_2 \left(2^{s+7}\frac{\|F_0\|_2}{\|F_0\|_0}\left(1 + \frac{1}{16\|F_0\|_2 A_2 \gamma}\right)\right)^{\frac{s-2}{\gamma}}\,. \tag{1.25}$$

(d) *If in addition either $0 < \gamma \le 1$ or one of the assumptions* **(H2)**, **(H3)** *is satisfied, then every solution F_t in part (c) (or generally in part (b)) satisfies a moment production estimate of exponential form:*

$$\int_{\mathbb{R}^N} e^{\alpha(t)\langle v\rangle^\gamma} \mathrm{d}F_t(v) \le 2\|F_0\|_0 \qquad \forall\, t > 0\,, \tag{1.26}$$

where

$$\alpha(t) = 2^{-s_0}\frac{\|F_0\|_0}{\|F_0\|_2}(1 - e^{-\beta t})\,, \quad \beta = 16\|F_0\|_2 A_2 \gamma > 0$$

and $1 < s_0 < \infty$ depends only on $b(\cdot)$ and γ.

It is possible to deduce from the previous theorem some more conventional moment estimates in exponential form where the constant in the argument of the exponential moment remains time-dependent:

Corollary 1.4. *Under the same assumptions on $B(z,\sigma)$ and the initial datum F_0 in Theorem 1.3, there exists a conservative measure weak solution F_t of Eq.* (1.1) *such that for any $0<s<\gamma$ and any $c>0$*

$$\int_{\mathbb{R}^N} e^{c\langle v\rangle^s}\mathrm{d}F_t(v)\le \left(e^{\alpha_s(t)}+2\right)\|F_0\|_0 \qquad \forall\, t>0$$

where

$$\alpha_s(t)=c\left(\frac{c}{\alpha(t)}\right)^{\frac{s}{\gamma-s}}\,.$$

Proof of Corollary 1.4. The proof of this Corollary is quite short and we can present it here. As a consequence of Theorem 1.3 there exists a conservative measure weak solution F_t of Eq. (1.1) such that F_t satisfies (1.26). For any $t>0$, by definition of $\alpha_s(t)$ and $0<s<\gamma$ we have

$$c\langle v\rangle^s>\alpha_s(t) \implies c\langle v\rangle^s=\left(\frac{\alpha_s(t)}{c\langle v\rangle^s}\right)^{\frac{\gamma-s}{s}}\alpha(t)\langle v\rangle^\gamma<\alpha(t)\langle v\rangle^\gamma\,.$$

Thus

$$\int_{\mathbb{R}^N} e^{c\langle v\rangle^s}\mathrm{d}F_t(v)=\int_{\{c\langle v\rangle^s\le\alpha_s(t)\}} e^{c\langle v\rangle^s}\mathrm{d}F_t(v)+\int_{\{c\langle v\rangle^s>\alpha_s(t)\}} e^{c\langle v\rangle^s}\mathrm{d}F_t(v)$$
$$\le e^{\alpha_s(t)}\|F_0\|_0+\int_{\{c\langle v\rangle^s>\alpha_s(t)\}} e^{\alpha(t)\langle v\rangle^\gamma}\mathrm{d}F_t(v)\le e^{\alpha_s(t)}\|F_0\|_0+2\|F_0\|_0\,.$$

□

Our second main result of this paper is

Theorem 1.5 (Uniqueness and stability estimates for locally integrable $b(\cdot)$)**.** *Let $B(z,\sigma)=|z|^\gamma b(\cos\theta)$ satisfy* **(H4)**. *Given any initial datum $F_0\in\mathcal{B}_2^+(\mathbb{R}^N)$ with $\|F_0\|_0\neq 0$, we have*

- (a) *Every conservative measure weak solution of Eq.* (1.1) *is a strong solution, while every measure strong solution of Eq.* (1.1) *is a measure weak solution.*
- (b) *Let F_t be a measure strong solution of Eq.* (1.1) *with the initial datum F_0 satisfying $\|F_t\|_2\le\|F_0\|_2$ for all $t\ge 0$. Then F_t in fact conserves the mass, momentum and energy.*
- (c) *There exists a unique conservative measure strong solution F_t of Eq.* (1.1) *such that $F_t|_{t=0}=F_0$. Therefore F_t satisfies the moment production estimates in Theorem 1.3 .*
- (d) *Let F_t be the unique conservative measure strong solutions of Eq.* (1.1) *with the initial datum F_0 and let G_t be a conservative measure strong solutions of Eq.* (1.1) *on the time interval $[\tau,\infty)$ with an initial datum $G_t|_{t=\tau}=G_\tau\in\mathcal{B}_2^+(\mathbb{R}^N)$ for some $\tau\ge 0$. Then:*
  - *If $\tau=0$, then*

$$\|F_t-G_t\|_2\le\Psi_{F_0}(\|F_0-G_0\|_2)e^{C(1+t)},\quad t\ge 0 \tag{1.27}$$

where Ψ_{F_0} is given by (1.22), *$C=\mathcal{R}(\gamma,A_0,A_2\,\|F_0\|_0,\|F_0\|_2)$ is an explicit positive continuous function on $(\mathbb{R}_{>0})^5$.*

– *If* $\tau > 0$*, then*

$$\|F_t - G_t\|_2 \le \|F_\tau - G_\tau\|_2 e^{c_\tau (t-\tau)}, \quad t \in [\tau, \infty) \tag{1.28}$$

where $c_\tau = 4A_0(\mathcal{K}_{2+\gamma}(F_0) + \|F_0\|_2)(1 + \frac{1}{\tau})$, $\mathcal{K}_{2+\gamma}(F_0)$ *is given in (1.25) with* $s = 2+\gamma$.

(e) *If* F_0 *is absolutely continuous with respect to the Lebesgue measure, i.e.* $\mathrm{d}F_0(v) = f_0(v)\mathrm{d}v$ *with* $0 \le f_0 \in L^1_2(\mathbb{R}^N)$*, then the unique conservative measure strong solution* F_t *with the initial datum* F_0 *is also absolutely continuous with respect to the Lebesgue measure:* $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v$ *for all* $t \ge 0$*, and* f_t *is the unique conservative mild solution of Eq.* (1.1) *with the initial datum* f_0.

(f) *If* F_0 *is not a Dirac mass and let* F_t *be the unique measure strong solution of Eq.* (1.1) *with the initial datum* F_0*, then there is a sequence* $\{f^n_t\}$ *of conservative* L^1*-solutions of Eq.* (1.1) *with initial data* $0 \le f^n_0 \in L^1_2(\mathbb{R}^N)$ *satisfying*

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f^n_0(v)\mathrm{d}v = \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}F_0(v)\,, \quad n = 1, 2, \dots \tag{1.29}$$

such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi(v) f^n_t(v)\mathrm{d}v = \int_{\mathbb{R}^N} \varphi(v)\mathrm{d}F_t(v) \qquad \forall\, \varphi \in C_b(\mathbb{R}^N), \quad \forall\, t \ge 0\,. \tag{1.30}$$

Remark 1.6. The trivial case $\|F_0\|_0 = 0$, i.e. $F_0 = 0$, is excluded from the above theorems since $F_0 = 0$ implies that $F_t \equiv 0$ is the unique conservative measure solution of Eq. (1.1).

Remark 1.7. An application of the estimate (1.28) for solutions with different initial times will be seen in our next paper concerning the rate of convergence to equilibrium.

Remark 1.8. In the second part of this work we shall prove the exponential convergence to equilibrium (for bounded angular function $b(\cdot)$): $\|F_t - M\|_0 \le Ce^{-ct}$ where M is the Maxwellian (Gaussian) with the same mass, momentum and energy as F_0 (assuming that F_0 is not a single Dirac mass and $\|F_0\|_0 \ne 0$), $C, c > 0$ are constants depending only on N, $b(\cdot)$, γ and the mass, momentum and energy of F_0. This result will allow us to improve the stability estimate (1.27) to be uniform in time:

$$\sup_{t\ge 0} \|F_t - G_t\|_2 \le \widetilde{\Psi}_{F_0}(\|F_0 - G_0\|_2)$$

for some explicit continuous function $\widetilde{\Psi}_{F_0}(r)$ on $[0, \infty)$ satisfying $\widetilde{\Psi}_{F_0}(0) = 0$.

1.5. **Strategy and plan of the paper.** We shall first in Section 2 prove some continuity and Lipschitz estimates on the collision operator Q in (weighted) total variation topology. In Section 3 we shall prove moment estimates, first on the kernel L_B and then on the collision operator Q, plus several technical lemmas on fractional binomial expansions, on the beta function and on some ODE estimates. After these two sections which remain purely at the level of functional inequalities, we shall start considering the time evolution problem and tackle the proof of the first main Theorem 1.3 in Section 4: the main step in the construction of weak measure solutions is based on an approximation argument with the help of the Mehler transform, and the moment estimates on the solutions will be proved with the help of the functional results in the previous section. Finally in Section 5 we shall prove the second main Theorem 1.5 by carefully revisiting the uniqueness estimates known for functions in the case of measures.

2. Regularity estimates on the collision operator

We shall prove in this section some continuity and Lipschitz estimates on the collision operator in the (weighted) total variation topology. It will be useful for defining measure weak solutions of Eq. (1.1) as we mentioned in Section 1, but also for proving weak convergence of approximate solutions, which leads to the existence of measure weak solutions. We start with a preliminary useful representation of the collision velocities.

2.1. **Representations of $\langle v'\rangle^2, \langle v'_*\rangle^2$.** We first begin this section with a preliminary technical computation.

For any $v, v_* \in \mathbb{R}^N$, let us define

$$\mathbf{h} = \frac{v+v_*}{|v+v_*|} \quad \text{for} \quad v+v_* \neq 0\,; \quad \mathbf{h} = \mathbf{e}_1 = (1,0,...,0) \quad \text{for} \quad v+v_* = 0$$

and recall that $\mathbf{n} = (v-v_*)/|v-v_*|$ when $v \neq v_*$ and $\mathbf{n} = \mathbf{e}_1$ else. By (1.3) we have

$$\begin{cases} \langle v'\rangle^2 := 1+|v'|^2 = \dfrac{\langle v\rangle^2+\langle v_*\rangle^2}{2} + \dfrac{|v+v_*||v-v_*|}{2}(\mathbf{h}\cdot\sigma) \\[2ex] \langle v'_*\rangle^2 := 1+|v'_*|^2 = \dfrac{\langle v\rangle^2+\langle v_*\rangle^2}{2} - \dfrac{|v+v_*||v-v_*|}{2}(\mathbf{h}\cdot\sigma)\,. \end{cases} \tag{2.1}$$

Let us also define the unit vector

$$\mathbf{j} = \frac{\mathbf{h}-(\mathbf{h}\cdot\mathbf{n})\mathbf{n}}{\sqrt{1-(\mathbf{h}\cdot\mathbf{n})^2}} \text{ for } |\mathbf{h}\cdot\mathbf{n}|<1 \text{ and } \mathbf{j}=\mathbf{e}_1 \text{ for } |\mathbf{h}\cdot\mathbf{n}|=1\,.$$

Then with the change of variables $\sigma = \cos\theta\mathbf{n} + \sin\theta\,\omega$, $\omega \in \mathbb{S}^{N-2}(\mathbf{n})$, we have

$$\mathbf{h}\cdot\sigma = (\mathbf{h}\cdot\mathbf{n})\cos\theta + \sqrt{1-(\mathbf{h}\cdot\mathbf{n})^2}\,\sin\theta\,(\mathbf{j}\cdot\omega)\,, \quad \omega\in\mathbb{S}^{N-2}(\mathbf{n})$$

so that we get another representation:

$$\begin{cases} \langle v'\rangle^2 = \langle v\rangle^2\cos^2\theta/2 + \langle v_*\rangle^2\sin^2\theta/2 + \sqrt{|v|^2|v_*|^2-(v\cdot v_*)^2}\,\sin\theta\,(\mathbf{j}\cdot\omega) \\ \langle v'_*\rangle^2 = \langle v\rangle^2\sin^2\theta/2 + \langle v_*\rangle^2\cos^2\theta/2 - \sqrt{|v|^2|v_*|^2-(v\cdot v_*)^2}\,\sin\theta\,(\mathbf{j}\cdot\omega)\,. \end{cases} \tag{2.2}$$

2.2. **Continuity estimate on the collision operator.**

Proposition 2.1 (Continuity of the collision operator)**.** *Let $B(z,\sigma)$ be given by* (1.4)-(1.5)-(1.6) *with $b(\cdot)$ satisfying* **(H0)**. *Then*

(I) *The function $(v,v_*) \mapsto L_B[\Delta\varphi](v,v_*)$ is continuous on $\mathbb{R}^N\times\mathbb{R}^N$ for all $\varphi \in C^2(\mathbb{R}^N)$.*

(II) *Let $B_n(z,\sigma) = \bar{B}_n(|z|,\cos\theta)$ satisfy* (1.5) *and*

$$\bar{B}_n(r,t) \nearrow \bar{B}(r,t) \quad (n\to\infty) \quad \forall\,(r,t)\in[0,\infty)\times(-1,1). \tag{2.3}$$

Then for any $\varphi\in C^2(\mathbb{R}^N)$ and any $0<R<\infty$

$$\sup_{|v|+|v_*|\le R} |L_{B_n}[\Delta\varphi](v,v_*) - L_B[\Delta\varphi](v,v_*)| \to 0 \quad (n\to\infty). \tag{2.4}$$

Moreover let $\varphi_n \in C^2(\mathbb{R}^N)$ satisfy

$$\lim_{n\to\infty}\varphi_n(v) = \varphi(v) \;\; \forall\, v\in\mathbb{R}^N\,; \quad \sup_{n\ge1}\sup_{|v|\le R}\sum_{|\alpha|\le2}|\partial^\alpha\varphi_n(v)| < \infty \quad \forall\,R<\infty\,. \tag{2.5}$$

Then

$$L_{B_n}[\Delta\varphi_n](v,v_*) \to L_B[\Delta\varphi](v,v_*) \quad (n\to\infty) \quad \forall\,(v,v_*)\in\mathbb{R}^N\times\mathbb{R}^N\,. \tag{2.6}$$

Proof of Proposition 2.1. Let us write

$$L_B[\Delta\varphi](v,v_*) = \int_0^{\pi} \bar{B}(|v-v_*|,\cos\theta)\sin^N\theta\, L[\Delta\varphi](v,v_*,\theta)\mathrm{d}\theta \tag{2.7}$$

where

$$L[\Delta\varphi](v,v_*,\theta) = \frac{1}{\sin^2\theta}\int_{\mathbb{S}^{N-2}(\mathbf{n})} \Delta\varphi\,\mathrm{d}\omega, \quad 0<\theta<\pi.$$

Recalling (1.11) we have

$$\sup_{0<\theta<\pi}|L[\Delta\varphi](v,v_*,\theta)| \le |\mathbb{S}^{N-2}|\left(\max_{|\xi|\le\sqrt{|v|^2+|v_*|^2}}|H_\varphi(\xi)|\right)|v-v_*|^2. \tag{2.8}$$

Part (I). For any $0<R<\infty$, consider decomposition

$$B(z,\sigma) = B(z,\sigma)\wedge R + (B(z,\sigma)-R)^+$$

where $x\wedge y = \min\{x,y\}$, $(x-y)^+ = \max\{x-y,0\}$. We have

$$L_B[\Delta\varphi](v,v_*) = L_{B\wedge R}[\Delta\varphi](v,v_*) + L_{(B-R)^+}[\Delta\varphi](v,v_*),$$
$$L_{B\wedge R}[\Delta\varphi](v,v_*) = \int_{\mathbb{S}^{N-1}}[B(v-v_*,\sigma)\wedge R]\Delta\varphi\,\mathrm{d}\sigma.$$

Fix any $(v_0,v_{*0})\in\mathbb{R}^N\times\mathbb{R}^N$. Applying (2.7)-(2.8) to $L_{(B-R)^+}[\Delta\varphi]$ and recalling the assumption (1.6) we have

$$\sup_{|v-v_0|^2+|v_*-v_{*0}|^2\le 1}|L_{(B-R)^+}[\Delta\varphi](v,v_*)| \le C_\varphi\int_0^{\pi}\Big(C_\gamma b(\cos\theta)-R\Big)^+\sin^N\theta\,\mathrm{d}\theta =: I_{\varphi,\gamma}(R)$$

where C_φ, C_γ are finite constants depending only on $\varphi,\gamma,v_0,v_{*0}$. Therefore

$$\begin{aligned} &|L_B[\Delta\varphi](v,v_*) - L_B[\Delta\varphi](v_0,v_{*0})| \\ &\quad\le |L_{B\wedge R}[\Delta\varphi](v,v_*) - L_{B\wedge R}[\Delta\varphi](v_0,v_{*0})| + I_{\varphi,\gamma}(R) \quad \forall\,|v-v_0|^2+|v_*-v_{*0}|^2\le 1. \end{aligned} \tag{2.9}$$

Let $(\Delta\varphi)_0 = \varphi(v_0{}') + \varphi(v_{*0}') - \varphi(v_0) - \varphi(v_{*0})$. Applying (2.7) to $L_{B\wedge R}[\Delta\varphi]$ and using the assumption (1.5) we have

$$\begin{aligned} &|L_{B\wedge R}[\Delta\varphi](v,v_*) - L_{B\wedge R}[\Delta\varphi](v_0,v_{*0})| \\ &\le C_\varphi|\mathbb{S}^{N-2}|\int_0^{\pi}\Big|\bar{B}(|v-v_*|,\cos\theta)\wedge R - \bar{B}(|v_0-v_{*0}|,\cos\theta)\wedge R\Big|\sin^{N-2}\theta\,\mathrm{d}\theta \\ &+R\int_{\mathbb{S}^{N-1}}\Big|\Delta\varphi-(\Delta\varphi)_0\Big|\,\mathrm{d}\sigma \to 0 \quad \text{as} \quad (v,v_*)\to(v_0,v_{*0}). \end{aligned}$$

Also by assumption $\int_0^{\pi} b(\cos\theta)\sin^N\theta\,\mathrm{d}\theta<\infty$ we have $I_{\varphi,\gamma}(R)\to 0$ as $R\to+\infty$. Thus from (2.9), by first letting $(v,v_*)\to(v_0,v_{*0})$ and then letting $R\to+\infty$, we obtain

$$\limsup_{(v,v_*)\to(v_0,v_{*0})}|L_B[\Delta\varphi](v,v_*) - L_B[\Delta\varphi](v_0,v_{*0})| = 0\,.$$

Part (II). By assumption (2.3) and (1.6) we have

$$\bar{B}_n(r,\cos\theta) \le \bar{B}_{n+1}(r,\cos\theta) \le \bar{B}(r,\cos\theta) \le (1+r^2)^{\gamma/2} b(\cos\theta)$$

which together with (1.5) implies that the functions

$$r \mapsto \int_0^\pi \bar{B}_n(r,\cos\theta)\sin^N\theta\,\mathrm{d}\theta, \quad r \mapsto \int_0^\pi \bar{B}(r,\cos\theta)\sin^N\theta\,\mathrm{d}\theta$$

are all continuous on $[0,\infty)$. Thus by first using (2.3) and dominated convergence and then using Dini's theorem we conclude that for any $0<R<\infty$

$$\int_0^\pi \Big(\bar{B}(r,\cos\theta) - \bar{B}_n(r,\cos\theta)\Big)\sin^N\theta\,\mathrm{d}\theta \to 0 \ \ (n\to 0) \ \text{ uniformly in } \ r\in[0,R].$$

Therefore applying (2.7)-(2.8) to $L_{B-B_n}[\Delta\varphi]$ we have, for any $0<R<\infty$,

$$\begin{aligned}&\sup_{|v|+|v_*|\le R}|L_B[\Delta\varphi](v,v_*) - L_{B_n}[\Delta\varphi](v,v_*)| = \sup_{|v|+|v_*|\le R}|L_{B-B_n}[\Delta\varphi](v,v_*)|\\ &\le C_{\varphi,R}\sup_{r\in[0,R]}\int_0^\pi \Big(\bar{B}(r,\cos\theta)-\bar{B}_n(r,\cos\theta)\Big)\sin^N\theta\,\mathrm{d}\theta \to 0 \quad (n\to\infty)\end{aligned}$$

where $C_{\varphi,R} = \sup\limits_{|\xi|\le R}|H_\varphi(\xi)|R^2$.

Finally for any $(v,v_*)\in\mathbb{R}^N\times\mathbb{R}^N$, using (2.5) and denoting $r=|v-v_*|$ we have by dominated convergence that

$$\begin{aligned}&|L_B[\Delta\varphi](v,v_*) - L_{B_n}[\Delta\varphi_n](v,v_*)| \le |L_B[\Delta(\varphi-\varphi_n)](v,v_*)| + |L_{B-B_n}[\Delta\varphi_n](v,v_*)|\\ &\le \int_0^\pi \bar{B}(r,\cos\theta)\sin^N\theta\ |L[\Delta(\varphi-\varphi_n)](v,v_*,\theta)|\,\mathrm{d}\theta\\ &+C\int_0^\pi \big(\bar{B}(r,\cos\theta)-\bar{B}_n(r,\cos\theta)\big)\sin^N\theta\,\mathrm{d}\theta \longrightarrow 0 \quad (n\to\infty)\end{aligned}$$

which concludes the proof. □

2.3. **A continuity estimate for product measures.** We shall now prove a continuity property for product measures which will prove useful for the construction of weak measure solutions.

Proposition 2.2 (A continuity property of product measures). *Let* $0\le s_j<\infty$, $\{\mu_j^n\}_{n=1}^\infty \subset \mathcal{B}^+_{s_j}(\mathbb{R}^{N_j})$, $\mu_j\in\mathcal{B}^+_0(\mathbb{R}^{N_j})$ *satisfy*

$$\sup_{n\ge 1}\|\mu_j^n\|_{s_j}<\infty, \quad j=1,2,\dots,k\,; \tag{2.10}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^{N_j}}\varphi_j\mathrm{d}\mu_j^n = \int_{\mathbb{R}^{N_j}}\varphi_j\mathrm{d}\mu_j\,, \quad \forall\,\varphi_j\in C_c^\infty(\mathbb{R}^{N_j}), \quad j=1,2,\dots,k\,. \tag{2.11}$$

Then

$$\mu_j\in\mathcal{B}^+_{s_j}(\mathbb{R}^{N_j}), \quad \|\mu_j\|_{s_j}\le\liminf_{n\to\infty}\|\mu_j^n\|_{s_j}, \quad j=1,2,\dots,k\,. \tag{2.12}$$

Moreover if $\Psi_n,\Psi\in C(\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\cdots\times\mathbb{R}^{N_k})$ *satisfy*

$$\lim_{|\mathbf{x}|\to\infty}\sup_{n\ge1}\frac{|\Psi_n(\mathbf{x})|}{\sum_{j=1}^k\langle x_j\rangle^{s_j}} = 0\,, \quad \lim_{n\to\infty}\sup_{|\mathbf{x}|\le R}|\Psi_n(\mathbf{x})-\Psi(\mathbf{x})| = 0 \tag{2.13}$$

for all $0 < R < \infty$*, where* $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \prod_{j=1}^k \mathbb{R}^{N_j}$*, then*

$$\lim_{n\to\infty} \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \Psi_n \mathrm{d}(\mu_1^n \otimes \mu_2^n \otimes \cdots \otimes \mu_k^n) = \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \Psi \mathrm{d}(\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k)\,. \tag{2.14}$$

Proof of Proposition 2.2. First (2.12) easily follows from Fatou's Lemma. Let us prove (2.14). Let

$$M = \sup_{n\geq 1} \{\|\mu_1^n\|_{s_1}\,,\, \|\mu_2^n\|_{s_2}\,,\, \dots\ ,\|\mu_k^n\|_{s_k}\}\,,$$
$$\nu^n = \mu_1^n \otimes \mu_2^n \otimes \cdots \otimes \mu_k^n\,, \quad \nu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_k\,.$$

By assumption on Ψ_n, Ψ, for any $\varepsilon > 0$ there exist $R \geq 1, n_\varepsilon \geq 1$ such that

$$|\Psi_n(\mathbf{x})|\ ,\ |\Psi(\mathbf{x})| < \varepsilon \sum_{j=1}^k \langle x_j \rangle^{s_j}, \quad \forall\, |\mathbf{x}| > R, \quad \forall\, n \geq n_\varepsilon; \tag{2.15}$$

$$|\Psi_n(\mathbf{x}) - \Psi(\mathbf{x})| < \varepsilon, \qquad \forall\, |\mathbf{x}| \leq 2kR\,, \quad \forall\, n \geq n_\varepsilon. \tag{2.16}$$

On the other hand, by polynomial approximation, there exists a polynomial $P(\mathbf{x})$ such that

$$|\Psi(\mathbf{x}) - P(\mathbf{x})| < \varepsilon \qquad \forall\, |\mathbf{x}| \leq 2kR\,. \tag{2.17}$$

Choose $\chi_j^R \in C_c^\infty(\mathbb{R}^{N_j})$ satisfying $0 \leq \chi_j^R(x_j) \leq 1$ on $\mathbb{R}^{N_j}$ and $\chi_j^R(x_j) = 1$ for $|x_j| \leq R$ and $\chi_j^R(x_j) = 0$ for $|x_j| \geq 2R$. If we write $P(\mathbf{x}) = \sum_{i=1}^m \prod_{j=1}^k P_{i,j}(x_j)$ where $m \in \mathbb{N}$ and $P_{i,j}(x_j)$ are polynomials in x_j, then

$$P(\mathbf{x}) \prod_{j=1}^k \chi_j^R(x_j) = \sum_{i=1}^m \prod_{j=1}^k \varphi_{i,j}(x_j)$$

where $\varphi_{i,j}(x_j) = P_{i,j}(x_j)\chi_j^R(x_j)$. Then consider the decomposition:

$$\begin{aligned}
&\int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \Psi_n \mathrm{d}\nu^n - \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \Psi \mathrm{d}\nu = \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \Psi_n \left(1 - \prod\nolimits_{j=1}^k \chi_j^R\right) \mathrm{d}\nu^n \\
&+ \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} (\Psi_n - \Psi) \prod\nolimits_{j=1}^k \chi_j^R \mathrm{d}\nu^n + \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} (\Psi - P) \prod\nolimits_{j=1}^k \chi_j^R \mathrm{d}\nu^n \\
&+ \left[\sum_{i=1}^m \prod_{j=1}^k \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \varphi_{i,j} \mathrm{d}\mu_j^n - \sum_{i=1}^m \prod_{j=1}^k \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} \varphi_{i,j} \mathrm{d}\mu_j\right] \\
&+ \int_{\prod_{j=1}^k \mathbb{R}^{N_j}} (P - \Psi) \prod\nolimits_{j=1}^k \chi_j^R \mathrm{d}\nu + \int_{\prod_{j=1}^k \mathbf{R}^{N_j}} \Psi \Big(\prod\nolimits_{j=1}^k \chi_j^R - 1\Big) \mathrm{d}\nu \\
&:= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4} + I_5 + I_6.
\end{aligned}$$

Since $1 - \prod_{j=1}^k \chi_j^R(x_j) = 0$ for all $|\mathbf{x}| \leq R$, and $\prod_{j=1}^k \chi_j^R(x_j) = 0$ for all $|\mathbf{x}| > 2kR$, it follows from (2.15)-(2.16)-(2.17) that for all $n \geq n_\varepsilon$

$$|I_{n,1}| + |I_6| \leq 2\varepsilon \int_{|\mathbf{x}|>R} \sum_{j=1}^k \langle x_j \rangle^{s_j} \mathrm{d}\nu^n \leq 2\varepsilon k M^k\,,$$

$$|I_{n,2}| + |I_{n,3}| + |I_5| \leq 2\varepsilon \int_{|\mathbf{x}|\leq 2kR} \mathrm{d}\nu^n + \varepsilon \int_{|\mathbf{x}|\leq 2kR} \mathrm{d}\nu \leq 3\varepsilon M^k\,.$$

For $I_{n,4}$, since $\varphi_{i,j}\in C_c^\infty(\mathbb{R}^{N_j})$, it follows from the assumption of the lemma that

$$I_{n,4}=\sum_{i=1}^{m}\Big(\prod_{j=1}^{k}\int_{\mathbb{R}^{N_j}}\varphi_{i,j}\mathrm{d}\mu_j^n-\prod_{j=1}^{k}\int_{\mathbb{R}^{N_j}}\varphi_{i,j}\mathrm{d}\mu_j\Big)\to 0\quad (n\to\infty).$$

Therefore

$$\limsup_{n\to\infty}\left|\int_{\prod_{j=1}^k\mathbb{R}^{N_j}}\Psi_n\mathrm{d}\nu^n-\int_{\prod_{j=1}^k\mathbb{R}^{N_j}}\Psi\mathrm{d}\nu\right|\le 5kM^k\varepsilon\,.$$

This proves (2.14) by letting $\varepsilon\to 0^+$. □

2.4. **Weighted Lipschitz regularity of the collision operator.** Let us prove some (weighted) Lipschitz properties on the collision operator acting on Borel measures, in the (weighted) total variation topology.

Proposition 2.3 (A weighted Lipschitz property on the collision operator)**.** *Let $B(z,\sigma)$ be given by* (1.4)-(1.5)-(1.6) *with $b(\cdot)$ satisfying* **(H4)**. *Then*

$$Q^\pm:\mathcal{B}_{s+\gamma}(\mathbb{R}^N)\times\mathcal{B}_{s+\gamma}(\mathbb{R}^N)\to\mathcal{B}_s(\mathbb{R}^N)\quad(s\ge 0)$$

are bounded and

$$\big\|Q^\pm(\mu,\nu)\big\|_s\le 2^{(s+\gamma)/2}A_0\left(\|\mu\|_{s+\gamma}\|\nu\|_0+\|\mu\|_0\|\nu\|_{s+\gamma}\right)\,, \tag{2.18}$$

$$\big\|Q^\pm(\mu,\mu)-Q^\pm(\nu,\nu)\big\|_s\le 2^{(s+\gamma)/2}A_0\left(\|\mu+\nu\|_{s+\gamma}\|\mu-\nu\|_0+\|\mu+\nu\|_0\|\mu-\nu\|_{s+\gamma}\right) \tag{2.19}$$

and hence

$$\|Q(\mu,\mu)-Q(\nu,\nu)\|_0\le 2^{1+(s+\gamma)/2}A_0\left(\|\mu+\nu\|_\gamma\|\mu-\nu\|_0+\|\mu+\nu\|_0\|\mu-\nu\|_\gamma\right)\,. \tag{2.20}$$

Finally for all $\mu\in\mathcal{B}_\gamma(\mathbb{R}^N)$ and all $\varphi\in C_b^2(\mathbb{R}^N)$, there holds

$$\langle Q(\mu,\mu),\,\varphi\rangle=\int_{\mathbb{R}^N}\varphi\mathrm{d}Q(\mu,\mu) \tag{2.21}$$

where the left-hand side of (2.21) *is defined in* (1.14).

Proof of Proposition 2.3. By elementary inequalities

$$\langle v'\rangle^s\le(\langle v\rangle^2+\langle v_*\rangle^2)^{s/2},\quad \left(1+|v-v_*|^2\right)^{\gamma/2}\le 2^{\gamma/2}\left(\langle v\rangle^2+\langle v_*\rangle^2\right)^{\gamma/2}$$

and the assumption on B we have for any $\varphi\in C_c(\mathbb{R}^N)$ with $\|\varphi\|_{L^\infty_{-s}}\le 1$

$$|\varphi(v')|B(v-v_*,\sigma)\le\langle v'\rangle^s\left(1+|v-v_*|^2\right)^{\gamma/2}b(\cos\theta)\le 2^{(s+\gamma)/2}(\langle v\rangle^{s+\gamma}+\langle v_*\rangle^{s+\gamma})b(\cos\theta)$$

and hence

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}L_B\left[|\varphi|\right](v,v_*)\mathrm{d}(|\mu|\otimes|\nu|)\le A_0 2^{(s+\gamma)/2}\left(\|\mu\|_{s+\gamma}\|\nu\|_0+\|\mu\|_0\|\nu\|_{s+\gamma}\right)\,,$$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}A(v-v_*)|\varphi(v)|\mathrm{d}(|\mu|\otimes|\nu|)\le A_0 2^{(s+\gamma)/2}\left(\|\mu\|_{s+\gamma}\|\nu\|_0+\|\mu\|_0\|\nu\|_{s+\gamma}\right)\,.$$

These imply (2.18). The inequality (2.19) follows from (2.18) and the following identities:

$$Q^\pm(\mu,\mu)-Q^\pm(\nu,\nu)=\frac{1}{2}Q^\pm(\mu+\nu,\mu-\nu)+\frac{1}{2}Q^\pm(\mu-\nu,\mu+\nu)\,.$$

Next recall $B(v-v_*,\sigma)=\bar{B}(|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma)$. By changing variables $\sigma\to-\sigma$, $v\leftrightarrow v_*$ and using Fubini's theorem we have

$$\int_{\mathbb{R}^N}\varphi\mathrm{d}Q^+(\mu,\mu)=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big(\int_{\mathbb{S}^{N-1}}B(v-v_*,\sigma)\Big(\varphi(v')+\varphi(v'_*)\Big)d\sigma\Big)\mathrm{d}\mu(v)\mathrm{d}\mu(v_*).$$

A similar symmetry for $\int_{\mathbb{R}^N}\varphi\mathrm{d}Q^-(\mu,\mu)$ is obvious. The difference of the two is equal to $\langle Q(\mu,\mu),\varphi\rangle$. This proves (2.21). □

3. Moment estimates on the collision operator

In this section we shall prove several inequalities on the moments of the collision operator which will be useful for the moment estimates of the weak measure solutions we shall construct.

3.1. **Analytical toolbox.** Let us first collect and prove some useful analytical results.

Lemma 3.1 (Fractional binomial expansion)**.** *Let $p\geq 1$ and $k_p=[(p+1)/2]$ the integer part of $(p+1)/2$. Then for all $x,y\geq 0$*

$$\sum_{k=0}^{k_p-1}\left(\begin{array}{c}p\\k\end{array}\right)\left(x^ky^{p-k}+x^{p-k}y^k\right)\leq(x+y)^p\leq\sum_{k=0}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\left(x^ky^{p-k}+x^{p-k}y^k\right)$$

where

$$\left(\begin{array}{c}p\\k\end{array}\right)=\frac{p(p-1)\cdots(p-k+1)}{k!},\quad k\geq 1\,;\quad\left(\begin{array}{c}p\\0\end{array}\right)=1\,.$$

Proof of Lemma 3.1. We refer to [7, Lemma 2] for the proof. □

Let $p\geq 1$ and $n\in\{1,2,\dots,[p]\}$. Then using Taylor's formula for the function $x\mapsto(1+x)^p$ one has

$$\sum_{k=0}^{n}\left(\begin{array}{c}p\\k\end{array}\right)x^k\leq(1+x)^p\quad\forall\,x\geq 0\,.$$

In particular

$$\sum_{k=0}^{n}\left(\begin{array}{c}p\\k\end{array}\right)\leq 2^p\,,\quad 1\leq n\leq p\,.\tag{3.1}$$

Let $\Gamma(x),\mathrm{B}(x,y)$ be the gamma and beta functions:

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t\,,\quad x>0\,;\quad\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t\,,\quad x,y>0\,.$$

It is well known that

$$\Gamma(x)\Gamma(y)=\Gamma(x+y)\mathrm{B}(x,y)\,,\quad\forall\,x,y>0\,.\tag{3.2}$$

Other relations that we shall also use are: For any integer $k\geq 1$ and for any real number $p\geq k$ we have

$$\left(\begin{array}{c}p\\k\end{array}\right)=\frac{\Gamma(p+1)}{\Gamma(p-k+1)\Gamma(k+1)}\,.\tag{3.3}$$

And

$$\mathrm{B}(x+1,y)+\mathrm{B}(x,y+1)=\mathrm{B}(x,y)\,,\quad x,y>0\,.\tag{3.4}$$

Lemma 3.2 (A stationary phase result)**.** *Let* $0<\alpha, R<\infty$*,* $g\in C([0,R])$ *and* $S\in C^1([0,R])$ *such that*

$$S(0)=0,\quad S'(x)<0 \qquad \forall\, x\in[0,R)\,.$$

Then for any $\lambda\geq 1$ *we have*

$$\int_0^R x^{\alpha-1}g(x)e^{\lambda S(x)}\mathrm{d}x=\Gamma(\alpha)\left(\frac{1}{-\lambda S'(0)}\right)^{\alpha}\Big(g(0)+o(1)\Big)$$

where $o(1)\to 0$ *as* $\lambda\to\infty$*.*

Proof of Lemma 3.2. This is classical stationary phase type of analysis, we omit the proof for the sake of conciseness of this paper. $\square$

Lemma 3.3 (An estimate on the beta function)**.** *Let* $p\geq 3$ *and* $k_p=[(p+1)/2]$*. Then*

$$\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\mathrm{B}(k,p-k)\leq 4\log p\,. \tag{3.5}$$

More generally for any $a>1$ *we have*

$$\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\mathrm{B}(ak,a(p-k))\leq C_a(ap)^{1-a}\,, \tag{3.6}$$

$$\sum_{k=0}^{k_p-1}\left(\begin{array}{c}p-2\\k\end{array}\right)\mathrm{B}(a(k+1),a(p-k-1))\leq C_a(ap)^{-a} \tag{3.7}$$

where $0<C_a<\infty$ *only depends on* a*.*

Proof of Lemma 3.3. Since $p\geq 3$ we have

$$\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\mathrm{B}(k,p-k)=\sum_{k=1}^{k_p}\frac{p}{k(p-k)}=\sum_{k=1}^{k_p}\Big(\frac{1}{k}+\frac{1}{p-k}\Big)\leq 2\sum_{k=1}^{k_p}\frac{1}{k}\leq 4\log p\,.$$

Now suppose $a>1$. Let

$$\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\mathrm{B}(ak,a(p-k))=I_a(p)+I_a(p,k_p)$$

where

$$I_a(p)=\sum_{k=1}^{k_p-1}\left(\begin{array}{c}p\\k\end{array}\right)\mathrm{B}(ak,a(p-k)),\quad I_a(p,k_p)=\left(\begin{array}{c}p\\k_p\end{array}\right)\mathrm{B}(ak_p,a(p-k_p)).$$

For the first term $I_a(p)$ we use the symmetry (w.r.t $x=1/2$) and Lemma 3.1 to get

$$\begin{aligned}I_a(p)&=\frac{1}{2}\int_0^1\frac{1}{x(1-x)}\Big\{\sum_{k=1}^{k_p-1}\left(\begin{array}{c}p\\k\end{array}\right)\Big(x^{ak}(1-x)^{a(p-k)}+x^{a(p-k)}(1-x)^{ak}\Big)\Big\}\mathrm{d}x\\&\leq\frac{1}{2}\int_0^1\frac{1}{x(1-x)}\Big\{\Big(x^a+(1-x)^a\Big)^p-x^{ap}-(1-x)^{ap}\Big\}\mathrm{d}x\\&=\int_0^{1/2}\frac{1}{x(1-x)}\Big\{\Big(x^a+(1-x)^a\Big)^p-x^{ap}-(1-x)^{ap}\Big\}\mathrm{d}x\,.\end{aligned}$$

Omitting the negative term $-x^{ap}$ we have

$$(x^a + (1-x)^a)^p - x^{ap} - (1-x)^{ap} \le p(x^a + (1-x)^a)^{p-1} x^a$$

so that

$$I_a(p) \le p \int_0^{1/2} x^{a-1} g_1(x) e^{pS(x)} \mathrm{d}x$$

where $g_1(x) = (1-x)^{-1}(x^a + (1-x)^a)^{-1}$ and $S(x) = \log(x^a + (1-x)^a)$, $x \in [0, 1/2]$. Since $g_1(0) = 1, S(0) = 0$ and

$$S'(0) = -a\,, \quad S'(x) = \frac{a\Big(x^{a-1} - (1-x)^{a-1}\Big)}{x^a + (1-x)^a} < 0 \qquad \forall\, x \in [0, 1/2)$$

(because $a > 1$) it follows from Lemma 3.2 that for all $p \ge 3$

$$I_a(p) \le C_a p \Gamma(a) \left(\frac{1}{pa}\right)^a = C_a (ap)^{1-a}\,.$$

For the second term $I_a(p, k_p)$ we use Stirling's formula

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{\frac{\theta_x}{12x}}\,, \quad \Gamma(x+1) = x\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\, e^{\frac{\theta_x}{12x}}\,, \quad x \ge 1$$

$(0 < \theta_x < 1)$ to compute

$$(3.8) \qquad I_a(p, k_p) = \frac{\Gamma(p+1)}{\Gamma(k_p+1)\Gamma(p-k_p+1)} \cdot \frac{\Gamma(ak_p)\Gamma(a(p-k_p))}{\Gamma(ap)}$$
$$\le e^{1/4} \frac{\sqrt{a}}{ap} \left(\frac{k_p}{p}\right)^{(a-1)k_p} \left(\frac{p-k_p}{p}\right)^{(a-1)(p-k_p)} \left(\frac{p}{k_p}\right)\left(\frac{p}{p-k_p}\right) \le C_a \frac{1}{ap}\left(\frac{1}{2}\right)^{(a-1)p}.$$

Here in the last inequality we used the simple estimates

$$\frac{p-1}{2} \le p - k_p \le \frac{p+1}{2}$$

for $p \ge 3$. This proves (3.6) because $a > 1$.

In order to prove (3.7) we consider again a decomposition

$$\sum_{k=0}^{k_p-1} \binom{p-2}{k} \mathrm{B}(a(k+1), a(p-k-1)) = J_a(p) + J_a(p, k_p)$$

where for the first term $J_a(p)$ we use that $k_p - 2 = [(p-1)/2] - 1 = k_{p-2} - 1$ and Lemma 3.1 to get

$$J_a(p) := \sum_{k=0}^{k_p-2} \binom{p-2}{k} \mathrm{B}(a(k+1), a(p-k-1))$$
$$= \frac{1}{2}\int_0^1 x^{a-1}(1-x)^{a-1} \sum_{k=0}^{k_{p-2}-1} \binom{p-2}{k} \Big(x^{ak}(1-x)^{a(p-2-k)} + x^{a(p-2-k)}(1-x)^{ak}\Big)\mathrm{d}x$$
$$\le \frac{1}{2}\int_0^1 x^{a-1}(1-x)^{a-1}\Big(x^a + (1-x)^a\Big)^{p-2}\mathrm{d}x = \int_0^{1/2} x^{a-1} g_2(x) e^{pS(x)}\mathrm{d}x$$

with $g_2(x) = (1-x)^{a-1}(x^a + (1-x)^a)^{-2}$. Since $a > 1$, it follows from Lemma 3.2 that

$$J_a(p) \le C_a \left(\frac{1}{ap}\right)^a .$$

For the second term $J_a(p, k_p)$ we use (3.8) to get

$$J_a(p,k_p) := \begin{pmatrix} p-2 \\ k_p-1 \end{pmatrix} \mathrm{B}(ak_p, a(p-k_p)) = \frac{(p-k_p)k_p}{p(p-1)} I_a(p,k_p) \le C_a \frac{1}{ap}\left(\frac{1}{2}\right)^{(a-1)p} .$$

Since $a > 1$, this proves the lemma. □

3.2. **An estimate of the angular cutoff reminder.**

Lemma 3.4. *Suppose $b(\cdot)$ satisfies the assumption* **(H0)**. *For all $p \ge 3$ we define*

$$\varepsilon_p := \frac{2}{A_2}\left|\mathbb{S}^{N-2}\right| \int_0^\pi \left\{ \int_0^1 t\left(1 - \frac{\sin^2\theta}{2}t\right)^{p-2} \mathrm{d}t \right\} b(\cos\theta)\sin^N\theta \,\mathrm{d}\theta \quad (\le 1) . \tag{3.9}$$

Then $\varepsilon_p \to 0$ $(p \to \infty)$. Furthermore, if either $0 < \gamma \le 1$ or **(H2)** *is satisfied, then*

$$p^{2-2/\gamma}\varepsilon_p \to 0 \qquad (p \to \infty) . \tag{3.10}$$

Proof of Lemma 3.4. Under the assumption **(H0)**, the convergence $\varepsilon_p \to 0\,(p \to \infty)$ is obvious and hence (3.10) holds for $0 < \gamma \le 1$. Suppose **(H2)** is satisfied, which means that $\nu = 2 - 2/\gamma \in (0,1)$ and $\theta \mapsto b(\cos\theta)\sin^{N-2\nu}\theta$ is integrable on $[0,\pi]$. For all $p \ge 3$ we have

$$p^\nu \varepsilon_p \le C \int_0^\pi \left\{ \int_0^1 \left((p-2)\frac{\sin^2\theta}{2}t\right)^\nu \left(1 - \frac{\sin^2\theta}{2}t\right)^{p-2} \mathrm{d}t \right\} b(\cos\theta)\sin^{N-2\nu}\theta \,\mathrm{d}\theta \tag{3.11}$$

where C depends only on N, A_2 and ν. Applying elementary estimates

$$0 \le (\lambda x)^\nu (1-x)^\lambda < 1 , \quad (\lambda x)^\nu (1-x)^\lambda \to 0 \quad (\lambda \to \infty) \quad \forall\, x \in [0,1]$$

to $\lambda = p-2$ and $x = \frac{\sin^2\theta}{2}t$ we conclude from (3.11) and the dominated convergence theorem that $p^\nu\varepsilon_p \to 0$ $(p \to \infty)$. □

Remark 3.5. It is easily calculated that if the assumption **(H4)** is satisfied, i.e. if $A_0 < \infty$, then $\varepsilon_p \le \frac{16A_0}{A_2}\frac{1}{p}$ for all $p \ge 3$, so that in case $0 < \gamma < 2$ we have $p^{2-2/\gamma}\varepsilon_p \le \frac{16A_0}{A_2}p^{1-2/\gamma}$.

3.3. **Moment estimates on the kernel L_B.** In this subsection we shall prove moment estimates on the kernel L_B as defined in (1.9).

Lemma 3.6. *Let $B(z,\sigma) = |z|^\gamma b(\cos\theta)$.*

(I) *Under the assumption* **(H0)** *we have for all* $p \geq 3$

$$
\begin{aligned}
(3.12)\quad & L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*) \\
& \leq -\frac{A_2}{4}\Big(\langle v\rangle^{2p+\gamma}+\langle v_*\rangle^{2p+\gamma}\Big)+\frac{A_2}{2}\Big(\langle v\rangle^{2p}\langle v_*\rangle^{\gamma}+\langle v_*\rangle^{2p}\langle v\rangle^{\gamma}\Big) \\
& +A_2\sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(\langle v\rangle^{2k+\gamma}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)+\gamma}\langle v_*\rangle^{2k}\Big) \\
& +A_2\sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)+\gamma}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k+\gamma}\Big) \\
& +2p(p-1)A_2\varepsilon_p\sum_{k=0}^{k_p-1}\left(\begin{array}{c} p-2 \\ k \end{array}\right)\Big(\langle v\rangle^{2(k+1)+\gamma}\langle v_*\rangle^{2(p-1-k)}+\langle v\rangle^{2(p-1-k)+\gamma}\langle v_*\rangle^{2(k+1)}\Big) \\
& +2p(p-1)A_2\varepsilon_p\sum_{k=0}^{k_p-1}\left(\begin{array}{c} p-2 \\ k \end{array}\right)\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)+\gamma}+\langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)+\gamma}\Big).
\end{aligned}
$$

(II) *Under the assumption* **(H3)** *which is rewritten in the form*

$$
(3.13)\quad \gamma=2,\quad 1<p_1<\infty\,,\quad A^*_{p_1}:=\left|\mathbb{S}^{N-2}\right|\left(\int_0^\pi [b(\cos\theta)]^{p_1}\sin^{N-2}\theta\,\mathrm{d}\theta\right)^{1/p_1}<\infty
$$

and let

$$
(3.14)\quad q_1=\frac{p_1}{p_1-1},\quad \eta=\frac{1}{2q_1}.
$$

Then

$$
\begin{aligned}
(3.15)\quad & L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*) \\
& \leq \frac{12A^*_{p_1}}{p^\eta}\sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k+1)}\langle v_*\rangle^{2k}\Big) \\
& +\frac{12A^*_{p_1}}{p^\eta}\sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k+1)}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2(k+1)}\Big) \\
& +\frac{A_0}{2}\langle v\rangle^{2p}\langle v_*\rangle^{2}+\frac{A_0}{2}\langle v_*\rangle^{2p}\langle v\rangle^{2}-\frac{A_0}{4}\langle v\rangle^{2(p+1)}-\frac{A_0}{4}\langle v_*\rangle^{2(p+1)}
\end{aligned}
$$

for all $p \geq \left(12A^*_{p_1}/A_0\right)^{2q_1}$.

Proof of Lemma 3.6.

Part (I) Let us write

$$
L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)=|v-v_*|^\gamma\left|\mathbb{S}^{N-2}\right|\int_0^\pi b(\cos\theta)\sin^N\theta\,L_p(v,v_*,\theta)\,\mathrm{d}\theta
$$

with

$$
L_p(v,v_*,\theta):=\frac{1}{\sin^2\theta\,|\mathbb{S}^{N-2}|}\int_{\mathbb{S}^{N-2}(\mathbf{k})}\Big(\langle v'\rangle^{2p}+\langle v'_*\rangle^{2p}-\langle v\rangle^{2p}-\langle v_*\rangle^{2p}\Big)\,\mathrm{d}\omega.
$$

We first prove that

$$
\begin{aligned}
(3.16)\qquad & L_p(v,v_*,\theta) \le -\frac{1}{2}\Big(\langle v\rangle^{2p} + \langle v_*\rangle^{2p}\Big) \\
& +\frac{1}{2}\sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2p-2k} + \langle v\rangle^{2p-2k}\langle v_*\rangle^{2k}\Big) \\
& +2p(p-1)\int_0^1 t\Big(1-\frac{\sin^2\theta}{2}t\Big)^{p-2}dt \\
& \times\sum_{k=0}^{k_p-1}\left(\begin{array}{c} p-2 \\ k \end{array}\right)\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)} + \langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)}\Big).
\end{aligned}
$$

To do this we denote the shorthand

$$E(\theta) = \langle v\rangle^2\cos^2\theta/2 + \langle v_*\rangle^2\sin^2\theta/2\,, \quad h = \sqrt{|v|^2|v_*|^2 - \langle v, v_*\rangle^2}\,.$$

Then by (2.2)

$$\langle v'\rangle^2 = E(\theta) + h\,\sin\theta\,(\mathbf{j}\cdot\omega)\,, \qquad \langle v'_*\rangle^2 = E(\pi-\theta) - h\,\sin\theta\,(\mathbf{j}\cdot\omega)\,.$$

By Taylor's formula we have

$$
\begin{aligned}
&\Big(E(\theta)\pm h\,\sin\theta\,(\mathbf{j}\cdot\omega)\Big)^p = \Big(E(\theta)\Big)^p \pm q\Big(E(\theta)\Big)^{p-1}h\,\sin\theta\,(\mathbf{j}\cdot\omega) \\
&+p(p-1)\int_0^1(1-t)\Big(E(\theta)\pm th\,\sin\theta\,\langle\mathbf{j},\omega\rangle\Big)^{p-2}\mathrm{d}t(h\,\sin\theta\,\langle\mathbf{j},\omega\rangle)^2\,.
\end{aligned}
$$

Look at the last term: We have for all $\theta\in(0,\pi), t\in[0,1]$

$$
\begin{aligned}
& E(\theta) + th\,\sin\theta\,|(\mathbf{j}\cdot\omega)| \le E(\theta) + \Big(E(\pi-\theta)\Big)\,t \\
& = \langle v\rangle^2 + \langle v_*\rangle^2 - \Big(E(\pi-\theta)\Big)(1-t) \\
& \le \Big(\langle v\rangle^2 + \langle v_*\rangle^2\Big)\Big(1-\frac{1-t}{2}\sin^2\theta\Big)
\end{aligned}
$$

where we used

$$E(\pi-\theta) \ge (\langle v\rangle^2 + \langle v_*\rangle^2)\min\{\cos^2\theta/2\,,\,\sin^2\theta/2\} \ge (\langle v\rangle^2 + \langle v_*\rangle^2)\frac{\sin^2\theta}{2}\,.$$

Since

$$\int_{\mathbb{S}^{N-2}(\mathbf{n})}(\mathbf{j}\cdot\omega)\mathrm{d}\omega = 0$$

it follows that

$$
\begin{aligned}
(3.17)\qquad & L_p(v,v_*,\theta) \le \frac{1}{\sin^2\theta}\left((E(\theta))^p + (E(\pi-\theta))^p - \langle v\rangle^{2p} - \langle v_*\rangle^{2p}\right) \\
& +2p(p-1)\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{p-2}h^2\int_0^1 t\left(1-\frac{\sin^2\theta}{2}t\right)^{p-2}\mathrm{d}t\,.
\end{aligned}
$$

We need to prove that for $p \ge 3$ and $k_p = [(p+1)/2]$

$$\frac{1}{\sin^2\theta}\Big((E(\theta))^p + (E(\pi-\theta))^p - \langle v\rangle^{2p} - \langle v_*\rangle^{2p}\Big) \tag{3.18}$$
$$\le -\frac{1}{2}\Big(\langle v\rangle^{2p} + \langle v_*\rangle^{2p}\Big) + \frac{1}{2}\sum_{k=1}^{k_p}\begin{pmatrix} p \\ k\end{pmatrix}\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2p-2k} + \langle v\rangle^{2p-2k}\langle v_*\rangle^{2k}\Big).$$

In fact using Lemma 3.1 we have

$$\begin{aligned}
&(E(\theta))^p + (E(\pi-\theta))^p \\
&\le \sum_{k=0}^{k_p}\begin{pmatrix} p \\ k\end{pmatrix}\Bigg(\left[\langle v\rangle^2\cos^2(\theta/2)\right]^k\left[\langle v_*\rangle^2\sin^2(\theta/2)\right]^{p-k} \\
&\qquad\qquad + \left[\langle v\rangle^2\cos^2(\theta/2)\right]^{p-k}\left[\langle v_*\rangle^2\sin^2(\theta/2)\right]^{k}\Bigg) \\
&+ \sum_{k=0}^{k_p}\begin{pmatrix} p \\ k\end{pmatrix}\Bigg(\left[\langle v\rangle^2\sin^2(\theta/2)\right]^k\left[\langle v_*\rangle^2\cos^2(\theta/2)\right]^{p-k} \\
&\qquad\qquad + \left[\langle v\rangle^2\sin^2(\theta/2)\right]^{p-k}\left[\langle v_*\rangle^2\cos^2(\theta/2)\right]^{k}\Bigg) \\
&\le \frac{\sin^2\theta}{2}\sum_{k=1}^{k_p}\begin{pmatrix} p \\ k\end{pmatrix}\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2p-2k} + \langle v\rangle^{2p-2k}\langle v_*\rangle^{2k}\Big) \\
&+\Big(\langle v\rangle^{2p} + \langle v_*\rangle^{2p}\Big)\Big(\cos^{2p}(\theta/2) + \sin^{2p}(\theta/2)\Big)
\end{aligned}$$

where we used the fact that $p \ge 3 \Longrightarrow p - k_p \ge 1$ so that

$$\cos^{2k}(\theta/2)\sin^{2p-2k}(\theta/2),\ \sin^{2k}(\theta/2)\cos^{2p-2k}(\theta/2) \le \frac{1}{4}\sin^2\theta$$

for all $k \in \{1, 2, ..., k_p\}$. Since $p \ge 3$ implies

$$\cos^{2p}(\theta/2) + \sin^{2p}(\theta/2) \le \cos^4(\theta/2) + \sin^4(\theta/2) = 1 - \frac{1}{2}\sin^2(\theta)$$

this gives (3.18).

Note that $h^2 \le \langle v\rangle^2\langle v_*\rangle^2$. Then using Lemma 3.1 again and recalling $k_p - 1 = k_{p-2} = [(p-1)/2]$ we have

$$\Big(\langle v\rangle^2 + \langle v_*\rangle^2\Big)^{p-2}h^2 \le \sum_{k=0}^{k_p-1}\begin{pmatrix} p-2 \\ k\end{pmatrix}\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)} + \langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)}\Big).$$

This together with (3.17)-(3.18) concludes the proof of (3.16).

Now using (3.16) and the definitions of $L_B[\Delta\varphi]$, A_2 and ε_p we obtain

$$L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)\le -\frac{A_2}{2}\Big(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\Big)|v-v_*|^{\gamma} \tag{3.19}$$

$$+\frac{A_2}{2}\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\ k\end{array}\right)\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2p-2k}+\langle v\rangle^{2p-2k}\langle v_*\rangle^{2k}\Big)|v-v_*|^{\gamma}$$

$$+p(p-1)A_2\varepsilon_p\sum_{k=0}^{k_p-1}\left(\begin{array}{c}p-2\\ k\end{array}\right)\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)}+\langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)}\Big)|v-v_*|^{\gamma}\,.$$

Next by $0<\gamma\le 2$ we have

$$|v-v_*|^{\gamma}\ge\frac{1}{2}\langle v\rangle^{\gamma}-\langle v_*\rangle^{\gamma}\,,\quad |v-v_*|^{\gamma}\ge\frac{1}{2}\langle v_*\rangle^{\gamma}-\langle v\rangle^{\gamma}\,. \tag{3.20}$$

Thus

$$\begin{aligned}&\left(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\right)|v-v_*|^{\gamma}=\langle v\rangle^{2p}|v-v_*|^{\gamma}+\langle v_*\rangle^{2p}|v-v_*|^{\gamma}\\ &\ge\langle v\rangle^{2p}\Big(\frac{1}{2}\langle v\rangle^{\gamma}-\langle v_*\rangle^{\gamma}\Big)+\langle v_*\rangle^{2p}\Big(\frac{1}{2}\langle v_*\rangle^{\gamma}-\langle v\rangle^{\gamma}\Big)\\ &=\frac{1}{2}\langle v\rangle^{2p+\gamma}+\frac{1}{2}\langle v_*\rangle^{2p+\gamma}-\langle v\rangle^{2p}\langle v_*\rangle^{\gamma}-\langle v_*\rangle^{2p}\langle v\rangle^{\gamma}\,.\end{aligned}$$

Since

$$|v-v_*|^{\gamma}\le 2(\langle v\rangle^{\gamma}+\langle v_*\rangle^{\gamma}) \tag{3.21}$$

it follows that

$$\begin{aligned}&\left(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k}\right)|v-v_*|^{\gamma}\\ &\le 2\left(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k}\right)(\langle v\rangle^{\gamma}+\langle v_*\rangle^{\gamma})\\ &=2\left(\langle v\rangle^{2k+\gamma}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)+\gamma}\langle v_*\rangle^{2k}\right)\\ &+2\left(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)+\gamma}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k+\gamma}\right)\,.\end{aligned}$$

And similarly

$$\begin{aligned}&\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)}+\langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)}\Big)|v-v_*|^{\gamma}\\ &\le 2\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)}+\langle v\rangle^{2(p-1-k)}\langle v_*\rangle^{2(k+1)}\Big)(\langle v\rangle^{\gamma}+\langle v_*\rangle^{\gamma})\\ &=2\Big(\langle v\rangle^{2(k+1)+\gamma}\langle v_*\rangle^{2(p-1-k)}+\langle v\rangle^{2(p-1-k)+\gamma}\langle v_*\rangle^{2(k+1)}\Big)\\ &+2\Big(\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-1-k)+\gamma}+\langle v\rangle^{2(p-1-k)+\gamma}\langle v_*\rangle^{2(k+1)+\gamma}\Big)\,.\end{aligned}$$

These together with (3.19) yield the estimate (3.12).

Part (II) For any $p\ge 1$ we have

$$\begin{aligned}&|v-v_*|^{-2}L_B[\Delta\langle\cdot\rangle^{2p}](v,v_*)=2\int_{\mathbb{S}^{N-1}}b(\cos\theta)\langle v'\rangle^{2p}\mathrm{d}\sigma-A_0\Big(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\Big)\\ &\le 2A^*_{p_1}\left(\frac{1}{|\mathbb{S}^{N-2}|}\int_{\mathbb{S}^{N-1}}\langle v'\rangle^{2pq_1}\mathrm{d}\sigma\right)^{1/q_1}-A_0\Big(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\Big)\end{aligned}$$

where we used Hölder's inequality. We have to prove that

$$\left(\frac{1}{|\mathbb{S}^{N-2}|}\int_{\mathbb{S}^{N-1}}\langle v'\rangle^{2pq_1}\mathrm{d}\sigma\right)^{1/q_1} \le \frac{3}{p^\eta}\Big(\langle v\rangle^2+\langle v_*\rangle^2\Big)^p. \tag{3.22}$$

To do this we denote $\lambda = pq_1\,(>1)$. Then using elementary inequalities

$$\left(\frac{1+x}{2}\right)^\lambda+\left(\frac{1-x}{2}\right)^\lambda \le \left(\frac{1+y}{2}\right)^\lambda+\left(\frac{1-y}{2}\right)^\lambda, \quad x,y\in[-1,1], \quad |x|\le|y|\,;$$

$$|v+v_*||v-v_*|\le \langle v\rangle^2+\langle v_*\rangle^2,$$

and the formula (2.1) we compute (recall that $N\ge 2$)

$$\begin{aligned}
&\frac{1}{|\mathbb{S}^{N-2}|}\int_{\mathbb{S}^{N-1}}\langle v'\rangle^{2\lambda}\mathrm{d}\sigma\\
&= \left(\langle v\rangle^2+\langle v_*\rangle^2\right)^\lambda\int_0^\pi \sin^{N-2}\theta\left(\frac12+\frac{|v+v_*||v-v_*|}{2(\langle v\rangle^2+\langle v_*\rangle^2)}\cos\theta\right)^\lambda\mathrm{d}\theta\\
&\le \left(\langle v\rangle^2+\langle v_*\rangle^2\right)^\lambda\int_0^\pi\left(\frac{1-\cos\theta}{2}\right)^\lambda\mathrm{d}\theta \le \Big(\langle v\rangle^2+\langle v_*\rangle^2\Big)^\lambda\sqrt{\frac{2\pi}{\lambda}}
\end{aligned}$$

where we used the well-known inequality

$$\int_0^{\pi/2}\sin^n\theta\,\mathrm{d}\theta<\sqrt{\frac{\pi}{2n}}$$

with $n=2[\lambda]$. This yields (3.22).

From this and using Lemma 3.1 we obtain that for all $p\ge 3$

$$\begin{aligned}
&|v-v_*|^{-2}L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)\le\frac{6A^*_{p_1}}{p^\eta}\left(\langle v\rangle^2+\langle v_*\rangle^2\right)^p-A_0\left(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\right)\\
&\le\frac{6A^*_{p_1}}{p^\eta}\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\Big(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k}\Big)-\Big(A_0-\frac{6A^*_{p_1}}{p^\eta}\Big)\Big(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\Big).
\end{aligned}$$

Since $p\ge\left(12A^*_{p_1}/A_0\right)^{2q_1}\iff 6A^*_{p_1}/p^\eta\le A_0/2$, it follows that

$$\begin{aligned}
&L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)\\
&\le\frac{6A^*_{p_1}}{p^\eta}\sum_{k=1}^{k_p}\left(\begin{array}{c}p\\k\end{array}\right)\left(\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k}\right)|v-v_*|^2\\
&-\frac{A_0}{2}\left(\langle v\rangle^{2p}+\langle v_*\rangle^{2p}\right)|v-v_*|^2\qquad\forall\,p\ge\left(12A^*_{p_1}/A_0\right)^{2q_1}.
\end{aligned}$$

Therefore as shown in the above using (3.20)-(3.21) with $\gamma=2$ we obtain (3.15). □

3.4. **Moment estimates on the collision operator.** We shall now deduce from the moment estimates on L_B in the previous Lemma 3.6 some moment estimates on the collision operator.

Lemma 3.7. *Let $B(z,\sigma)=|z|^\gamma b(\cos\theta)$, $\mu\in\mathcal{B}^+_s(\mathbb{R}^N)$ with $\|\mu\|_0\ne 0$, $s\ge\gamma+2p$, $0<\gamma\le 2$, and $p\ge 3$.*

(I) *If $b(\cos\theta)$ satisfies the assumption* **(H0)**, *then*

$$\left\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\right\rangle \le 2^{2p+1}A_2\|\mu\|_2\|\mu\|_{2p} - \frac{1}{4}A_2\|\mu\|_0\|\mu\|_{2p+\gamma}\,. \tag{3.23}$$

Furthermore if $0<\gamma<2$, then

$$\begin{aligned} &\frac{\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\rangle}{\Gamma(q)\|\mu\|_0} \\ &\le \Big(C_a q^{2-a} + C_a q^{3-a}\varepsilon_p\Big)A_2\|\mu\|_0 Z_p^* + \frac{1}{2}\|\mu\|_2 A_2 Z_q - \frac{q}{16}A_2\|\mu\|_0 Z_q^{1+\frac{1}{q}} \end{aligned} \tag{3.24}$$

where $q=ap$, $a=2/\gamma$,

$$Z_q = \frac{\|\mu\|_{\gamma q}}{\Gamma(q)\|\mu\|_0}, \qquad Z_p^* = \max_{k\in\{1,2,\dots,k_p\}}\{Z_{ak+1}Z_{a(p-k)}\,,\, Z_{ak}Z_{a(p-k)+1}\} \tag{3.25}$$

and the constant $0<C_a<\infty$ only depends on a.

(II) *If $\gamma=2$ and $b(\cos\theta)$ satisfies* **(H3)** *which is rewritten as in* (3.13), *and let p_1, q_1, η be given in* (3.13)-(3.14), *then*

$$\begin{aligned} &\frac{\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\rangle}{\Gamma(p)\|\mu\|_0} \\ &\le 48A^*_{p_1}p^{1-\eta}(\log p)\|\mu\|_0\widetilde{Z}_p^* + \Big(12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4}\Big)\|\mu\|_2 Z_p - \frac{p}{16}A_0\|\mu\|_0 Z_p^{1+\frac{1}{p}} \end{aligned} \tag{3.26}$$

for all $p\ge \big(12A^*_{p_1}/A_0\big)^{2q_1}$, where

$$Z_p = \frac{\|\mu\|_{2p}}{\Gamma(p)\|\mu\|_0}, \qquad \widetilde{Z}_p^* = \max_{k\in\{1,\,2,\dots,\,k_p\}} Z_{k+1}Z_{p-k}\,. \tag{3.27}$$

Proof of Lemma 3.7. By replacing μ with $\mu/\|\mu\|_0$ we can assume that $\|\mu\|_0=1$.
Part (I). By part (I) of Lemma 3.6 we have

$$\begin{aligned} \left\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\right\rangle &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)\mathrm{d}\mu(v)\mathrm{d}\mu(v_*) \\ &\le A_2\sum_{k=1}^{k_p}\binom{p}{k}\left(\|\mu\|_{2k+\gamma}\|\mu\|_{2(p-k)} + \|\mu\|_{2k}\|\mu\|_{2(p-k)+\gamma}\right) \\ &+2p(p-1)A_2\varepsilon_p\sum_{k=0}^{k_p-1}\binom{p-2}{k}\left(\|\mu\|_{2(k+1)+\gamma}\|\mu\|_{2(p-1-k)} + \|\mu\|_{2(k+1)}\|\mu\|_{2(p-1-k)+\gamma}\right) \\ &+\frac{A_2}{2}\|\mu\|_{2p}\|\mu\|_\gamma - \frac{A_2}{4}\|\mu\|_{2p+\gamma}\,. \end{aligned} \tag{3.28}$$

Using Hölder's inequality we have (for $s>2$)

$$\|\mu\|_r \le \|\mu\|_2^{\frac{s-r}{s-2}}\|\mu\|_s^{\frac{r-2}{s-2}}, \quad 2\le r\le s \tag{3.29}$$

from which we obtain for all $s_1, s_2\ge 2$ satisfying $s_1+s_2\le 2p+2$

$$\|\mu\|_{s_1}\|\mu\|_{s_2} \le \|\mu\|_2^{\frac{2p-s_1+2p-s_2}{2p-2}}\|\mu\|_{2p}^{\frac{s_1+s_2-4}{2p-2}} \le \|\mu\|_2\|\mu\|_{2p}$$

where we used $\|\mu\|_2 \le \|\mu\|_{2p}$. Thus

$$\begin{aligned}
\left\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\right\rangle &\le 4A_2\left\{\sum_{k=1}^{k_p}\binom{p}{k} + 2p(p-1)\sum_{k=0}^{k_p-1}\binom{p-2}{k}\right\}\|\mu\|_2\|\mu\|_{2p} \\
&+ \frac{A_2}{2}\|\mu\|_2\|\mu\|_{2p} - \frac{A_2}{4}\|\mu\|_{2p+\gamma} \\
&\le 4A_2\left(2^p - 1 + p(p-1)2^{p-1}\right)\|\mu\|_2\|\mu\|_{2p} + \frac{A_2}{2}\|\mu\|_2\|\mu\|_{2p} - \frac{A_2}{4}\|\mu\|_{2p+\gamma} \\
&\le 2^{2p+1}A_2\|\mu\|_2\|\mu\|_{2p} - \frac{A_2}{4}\|\mu\|_{2p+\gamma}
\end{aligned}$$

which proves (3.23) for $\|\mu\|_0 = 1$, where we used the inequality (3.1) and

$$2^p + p(p-1)2^{p-1} \le 2^{2p-1}\,, \quad p \ge 3\,.$$

Now suppose that $0 < \gamma < 2$. This implies $a = 2/\gamma > 1$. Recall definitions of Z_q and Z_p^* in (3.25). Then applying (3.2) and (3.4) we compute for all $k \in \{1, 2, ..., k_p\}$

$$\begin{aligned}
&\|\mu\|_{2k+\gamma}\|\mu\|_{2(p-k)} + \|\mu\|_{2k}\|\mu\|_{2(p-k)+\gamma} \\
&= \|\mu\|_{\gamma(ak+1)}\|\mu\|_{\gamma a(p-k)} + \|\mu\|_{\gamma ak}\|\mu\|_{\gamma(a(p-k)+1)} \\
&= Z_{ak+1}Z_{a(p-k)}\Gamma(ak+1)\Gamma(a(p-k)) + Z_{ak}Z_{a(p-k)+1}\Gamma(ak)\Gamma(a(p-k)+1) \\
&\le Z_p^*\Gamma(ap+1)\Big(\mathrm{B}(ak+1, a(p-k)) + \mathrm{B}(ak, a(p-k)+1)\Big) \\
&= Z_p^*\Gamma(q+1)\mathrm{B}(ak, a(p-k))\,,
\end{aligned}$$

and for all $k \in \{0, 1, ..., k_p - 1\}$

$$\begin{aligned}
&\|\mu\|_{2(k+1)+\gamma}\|\mu\|_{2(p-1-k)} + \|\mu\|_{2(k+1)}\|\mu\|_{2(p-1-k)+\gamma} \\
&= Z_{a(k+1)+1}Z_{a(p-k-1)}\Gamma(a(k+1)+1)\Gamma(a(p-1-k)) \\
&+ Z_{a(k+1)}Z_{a(p-1-k)+1}\Gamma(a(k+1))\Gamma(a(p-1-k)+1) \\
&\le Z_p^*\Gamma(q+1)\mathrm{B}(a(k+1), a(p-1-k))\,.
\end{aligned}$$

This together with $\Gamma(q+1)/\Gamma(q) = q$ and Lemma 3.3 gives from (3.28) that

$$\begin{aligned}
\frac{\left\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\right\rangle}{\Gamma(q)} &\le Z_p^* q A_2 \sum_{k=1}^{k_p}\binom{p}{k}\mathrm{B}(ak, a(p-k)) \\
&+ Z_p^* 2qp(p-1)A_2\varepsilon_p \sum_{k=0}^{k_p-1}\binom{p-2}{k}\mathrm{B}(a(k+1), a(p-1-k)) \\
&+ \frac{A_2\|\mu\|_2}{2}Z_q - \frac{A_2}{4}\frac{\|\mu\|_{2p+\gamma}}{\Gamma(q)} \\
&\le Z_p^* A_2 C_a q^{2-a} + Z_p^* A_2 C_a q^{3-a}\varepsilon_p + \frac{A_2\|\mu\|_2}{2}Z_q - \frac{A_2}{4}\frac{\|\mu\|_{2p+\gamma}}{\Gamma(q)}\,.
\end{aligned} \tag{3.30}$$

For the negative term we use Hölder's inequality, $\|\mu\|_0 = 1$, and $q = ap = \frac{2p}{\gamma}$ to get

$$\|\mu\|_{2p+\gamma} \ge \|\mu\|_{2p}^{1+\frac{\gamma}{2p}} = \|\mu\|_{\gamma q}^{1+\frac{1}{q}}$$

and so

$$\frac{\|\mu\|_{2p+\gamma}}{\Gamma(q)} \ge \Gamma(q)^{\frac{1}{q}}\left(\frac{\|\mu\|_{\gamma q}}{\Gamma(q)}\right)^{1+\frac{1}{q}} = \Gamma(q)^{\frac{1}{q}} Z_q^{1+\frac{1}{q}} \ge \frac{q}{4} Z_q^{1+\frac{1}{q}} \tag{3.31}$$

where we have used the inequality $\Gamma(q)^{\frac{1}{q}} \ge q/4$. Thus (3.25) (with $\|\mu\|_0 = 1$) follows from (3.30).

Part (II). In this case we have $\gamma = 2$, i.e. $a = 1$ so that $q = p$ and hence (3.31) becomes

$$\frac{\|\mu\|_{2(p+1)}}{\Gamma(p)} \ge \frac{p}{4} Z_p^{1+\frac{1}{p}}.$$

By part (II) of Lemma 3.6 we have, as shown above, that (the special term $\|\mu\|_{2k}\|\mu\|_{2(p-k+1)}$ for $k = 1$ in the sum should be treated separately)

$$\begin{aligned}
&\frac{\langle Q(\mu,\mu),\, \langle\cdot\rangle^{2p}\rangle}{\Gamma(p)} = \frac{1}{2\Gamma(p)} \iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B\left[\Delta\langle\cdot\rangle^{2p}\right](v,v_*)\mathrm{d}\mu(v)\mathrm{d}\mu(v_*)\\
&\le \frac{1}{\Gamma(p)}\cdot\frac{12A^*_{p_1}}{p^\eta}\sum_{k=1}^{k_p}\binom{p}{k}\left(\|\mu\|_{2(k+1)}\|\mu\|_{2(p-k)} + \|\mu\|_{2k}\|\mu\|_{2(p-k+1)}\right)\\
&+\frac{1}{4\Gamma(p)}A_0\|\mu\|_2\|\mu\|_{2p} - \frac{A_0}{4}\frac{\|\mu\|_{2(p+1)}}{\Gamma(p)}\\
&= \frac{1}{\Gamma(p)}\cdot\frac{12A^*_{p_1}}{p^\eta}\sum_{k=2}^{k_p}\binom{p}{k}\left(\|\mu\|_{2(k+1)}\|\mu\|_{2(p-k)} + \|\mu\|_{2k}\|\mu\|_{2(p-k+1)}\right)\\
&+\frac{1}{\Gamma(p)}\cdot\frac{12A^*_{p_1}}{p^\eta}\binom{p}{1}\|\mu\|_4\|\mu\|_{2(p-1)} + \frac{1}{\Gamma(p)}\cdot\frac{12A^*_{p_1}}{p^\eta}\binom{p}{1}\|\mu\|_2\|\mu\|_{2p}\\
&+\frac{A_0}{4}\|\mu\|_2\frac{\|\mu\|_{2p}}{\Gamma(p)} - \frac{A_0}{4}\frac{\|\mu\|_{2(p+1)}}{\Gamma(p)}\\
&\le \widetilde{Z}^*_p\frac{12A^*_{p_1}}{p^\eta}\cdot p\sum_{k=2}^{k_p}\binom{p}{k}\mathrm{B}(k,p-k) + \widetilde{Z}^*_p\cdot\frac{12A^*_{p_1}}{p^\eta}p\binom{p}{1}\mathrm{B}(2,p-1)\\
&+\Big(12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4}\Big)\|\mu\|_2 Z_p - \frac{p}{16}A_0 Z_p^{1+\frac{1}{p}}\\
&\le \widetilde{Z}^*_p\frac{12A^*_{p_1}}{p^\eta}\cdot p\sum_{k=1}^{k_p}\binom{p}{k}\mathrm{B}(k,p-k) + \Big(12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4}\Big)\|\mu\|_2 Z_p - \frac{p}{16}A_0 Z_p^{1+\frac{1}{p}}\\
&\le 48A^*_{p_1}p^{1-\eta}(\log p)\widetilde{Z}^*_p + \Big(12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4}\Big)\|\mu\|_2 Z_p - \frac{p}{16}A_0 Z_p^{1+\frac{1}{p}}
\end{aligned}$$

where in the last inequality we used Lemma 3.3. This proves (3.26) for $\|\mu\|_0 = 1$. □

3.5. **An ODE comparison inequality.** Finally we shall conclude this section by proving an ODE comparison inequality which will be useful for proving moment production estimates.

Lemma 3.8. *Given any $A > 0, B > 0, \varepsilon > 0$, we have:*

(I) *The function*

$$Y(t)=\left(\frac{A}{B(1-e^{-\varepsilon At})}\right)^{1/\varepsilon}, \quad t>0$$

is the unique positive C^1-solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t)=AY(t)-BY(t)^{1+\varepsilon}, \quad t>0; \quad Y(0+)=\infty.$$

(II) *Let $u(t)$ be a non-negative function in $(0,\infty)$ with the properties that u is absolutely continuous on every bounded closed subinterval of $(0,\infty)$ and*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right)1_{\{u(t)>Y(t)\}}\le\left(Au(t)-Bu(t)^{1+\varepsilon}\right)1_{\{u(t)>Y(t)\}} \quad \text{a.e.} \quad t\in(0,\infty).$$

Then $u(t)\le Y(t)$ for all $t\in(0,\infty)$.

Proof of Lemma 3.8. Part (I) is obvious. To prove part (II) we use the assumption on u and notice that the function $x\mapsto Bx^{1+\varepsilon}-Ax$ is increasing in $((A/B)^{1/\varepsilon},\infty)$ and $Y(t)>(A/B)^{1/\varepsilon}$. Then it follows from the assumption of the lemma that

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t)-\frac{\mathrm{d}}{\mathrm{d}t}Y(t)\right)1_{\{u(t)>Y(t)\}}\\ &\le\Big(BY(t)^{1+\varepsilon}-AY(t)-Bu(t)^{1+\varepsilon}+Au(t)\Big)1_{\{u(t)>Y(t)\}}\le 0 \quad \text{a.e.} \quad t\in(0,\infty).\end{aligned}$$

Thus by the absolute continuity of u we have for any $t>t_*>0$

$$\begin{aligned}&(u(t)-Y(t))^+\\ &=(u(t_*)-Y(t_*))^++\int_{t_*}^{t}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}u(\tau)-\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\right)1_{\{u(\tau)>Y(\tau)\}}\mathrm{d}\tau\le(u(t_*)-Y(t_*))^+.\end{aligned}$$

From this we see it is enough to prove that for any $t>0$ there is $t_*\in(0,t)$ such that $u(t_*)\le Y(t_*)$. Otherwise there were $t_0>0$ such that $u(t)>Y(t)$ for all $t\in(0,t_0)$. By assumption on u, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t)\le Au(t)-Bu(t)^{1+\varepsilon} \quad \text{a.e.} \quad t\in(0,t_0).$$

On the other hand, from the lower bound $Y(t)>(A/B)^{1/\varepsilon}$ we see that the function $t\mapsto u^{-\varepsilon}(t)$ is absolutely continuous on every closed subinterval of $(0,t_0]$. We then compute for a.e. $t\in(0,t_0)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u^{-\varepsilon}(t))\ge-\varepsilon Au^{-\varepsilon}(t)+\varepsilon B$$

and hence for any $0<\tau<t_0$ we have by the absolute continuity of $t\mapsto u^{-\varepsilon}(t)e^{\varepsilon At}$ on $[\tau,t_0]$ that

$$u^{-\varepsilon}(t)e^{\varepsilon At}\ge u^{-\varepsilon}(\tau)e^{\varepsilon A\tau}+\frac{B(e^{\varepsilon At}-e^{\varepsilon A\tau})}{A}, \quad \forall\, t\in[\tau,t_0].$$

Omitting the positive term $u^{-\varepsilon}(\tau)e^{\varepsilon A\tau}$ and letting $\tau\to 0+$ leads to

$$u^{-\varepsilon}(t)e^{\varepsilon At}\ge\frac{B(e^{\varepsilon At}-1)}{A}, \quad \forall\, t\in(0,t_0]$$

i.e.

$$u(t) \le \left(\frac{A}{B(1-e^{-\varepsilon At})}\right)^{1/\varepsilon} = Y(t) \quad \forall\, t \in (0, t_0]$$

which contradicts the assertion "$u(t) > Y(t)$ for all $t \in (0, t_0)$". This prove the existence of $t_* \in (0, t)$ for all $t > 0$ and therefore concludes the proof of the lemma. □

4. Construction of weak measure solutions: Proof of Theorem 1.3

For notation convenience we denote

$$\int_{\mathbb{R}^N} \varphi \mathrm{d}F_t = \int_{\mathbb{R}^N} \varphi(v)\mathrm{d}F_t(v), \quad \text{etc.}$$

And note that if F_t is a measure weak solution of Eq. (1.1), then for any $\varphi \in C_b^2(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \varphi \mathrm{d}F_t = \int_{\mathbb{R}^N} \varphi \mathrm{d}F_{t_0} + \int_{t_0}^{t} \langle Q(F_\tau, F_\tau),\, \varphi\rangle \,\mathrm{d}\tau \quad \forall\, t > t_0 > 0\,. \tag{4.1}$$

Our proofs of the parts (a)-(b)-(c)-(d) of Theorem 1.3 are contained in the following three steps.

Step 1. A priori estimates for measure weak solutions. We first prove part (b) and moreover we prove that the solution F_t in part (b) satisfies that for any $s \ge 0$ and any $\varphi \in L^\infty_{-s} \cap C^2(\mathbb{R}^N)$,

$$t \mapsto \langle Q(F_t, F_t),\, \varphi\rangle \quad \text{is continuous in} \quad (0, \infty) \tag{4.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \varphi \mathrm{d}F_t = \langle Q(F_t, F_t),\, \varphi\rangle \quad \forall\, t > 0\,. \tag{4.3}$$

And these integrals are absolutely convergent for any $t > 0$. Then we prove that F_t satisfies the moment production estimates in parts (c) and (d) of Theorem 1.3.

Now let F_t satisfy the assumptions in part (b). Recall that F_t already conserves the mass as mentioned in Definition 1.1. Therefore the assumption $\|F_t\|_2 \le \|F_0\|_2\, (\forall\, t > 0)$ is equivalent to the energy inequality

$$\int_{\mathbb{R}^N} |v|^2 \mathrm{d}F_t(v) \le \int_{\mathbb{R}^N} |v|^2 \mathrm{d}F_0(v) \quad \forall\, t > 0. \tag{4.4}$$

Since our test function space for defining measure weak solutions is only $C_b^2(\mathbb{R}^N)$, we need a truncation-mollification approximation. Let $\chi \in C_c^\infty(\mathbb{R}^N)$ satisfy $0 \le \chi \le 1$ on $\mathbb{R}^N$ and $\chi(v) = 1$ for $|v| \le 1$, $\chi(v) = 0$ for $|v| \ge 2$. Given any $s \ge 0$ and any $\varphi \in L^\infty_{-s} \cap C^2(\mathbb{R}^N)$, let $\varphi_n(v) := \varphi(v)\chi(v/n)$. It is easily seen that $\varphi_n \in C_c^2(\mathbb{R}^N) \subset C_b^2(\mathbb{R}^N)$ and their Hessian matrices satisfy

$$\sup_{n\ge 1} |H_{\varphi_n}(v)| \le C_\varphi \langle v\rangle^s\,.$$

Thus by (1.11) we have for any $s_1 > s + 2 + \gamma$

$$\sup_{n\ge 1} \frac{|L_B\left[\Delta\varphi_n\right](v, v_*)|}{\langle v\rangle^{s_1} + \langle v_*\rangle^{s_1}} \le C_\varphi A_2 \frac{(\langle v\rangle^s + \langle v_*\rangle^s)|v - v_*|^{2+\gamma}}{\langle v\rangle^{s_1} + \langle v_*\rangle^{s_1}} \to 0$$

as $|v|^2+|v_*|^2\to\infty$, and by part (II) of Proposition 2.1, we deduce

$$\lim_{n\to\infty} L_B\left[\Delta\varphi_n\right](v,v_*) = L_B\left[\Delta\varphi\right](v,v_*) \qquad \forall\,(v,v_*)\in\mathbb{R}^N\times\mathbb{R}^N\,.$$

Thus by (4.1), the assumption (1.23) and the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_{t_0}^{t}\langle Q(F_\tau,F_\tau),\varphi_n\rangle\,\mathrm{d}\tau = \int_{t_0}^{t}\langle Q(F_\tau,F_\tau),\varphi\rangle\,\mathrm{d}\tau \quad \forall\, t>t_0>0$$

and thus (4.1) holds for all $\varphi\in\bigcup_{s\ge 0}L^\infty_{-s}\cap C^2(\mathbb{R}^N)$.

Since $\psi_j(v)=v_j$, $j=1,\dots,N$, and $\psi(v)=|v|^2$ belong to $L^\infty_{-2}\cap C^2(\mathbb{R}^N)$ and $\Delta\psi_j=\Delta\psi=0$, it follows from (4.1) that F_t conserves the momentum and energy in the *open* interval $(0,\infty)$. Therefore in order to prove the conservation of momentum and energy in the *closed* interval $[0,\infty)$, we only have to prove that

$$\lim_{t\to 0^+}\int_{\mathbb{R}^N} v_j\mathrm{d}F_t(v)=\int_{\mathbb{R}^N} v_j\mathrm{d}F_0(v), \quad \lim_{t\to 0^+}\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_t(v)=\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_0(v) \tag{4.5}$$

for $j=1,2,\dots,N$.

Let $\chi(v)$ be given above and let $\varepsilon>0$. Then $v\mapsto v_j\chi(\varepsilon v), v\mapsto |v|^2\chi(\varepsilon v)$ belong to $C^2_c(\mathbb{R}^N)\subset C^2_b(\mathbb{R}^N)$ so that, by definition of measure weak solutions, the functions

$$t\mapsto \int_{\mathbb{R}^N} v_j\chi(\varepsilon v)\mathrm{d}F_t(v) \quad \text{ and } \quad t\mapsto\int_{\mathbb{R}^N}|v|^2\chi(\varepsilon v)\mathrm{d}F_t(v)$$

are all continuous on $[0,\infty)$. Since

$$|v_j - v_j\chi(\varepsilon v)|\le |v|1_{\{|v|\ge 1/\varepsilon\}}\le \varepsilon|v|^2$$

and

$$C:=\sup_{t\ge 0}\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_t(v)\le\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_0(v)<\infty,$$

it follows that

$$\int_{\mathbb{R}^N} v_j\mathrm{d}F_t(v)=\int_{\mathbb{R}^N} v_j\chi(\varepsilon v)\mathrm{d}F_t(v)+O(\varepsilon) \qquad \forall\, t\ge 0$$

where $|O(\varepsilon)|\le C\varepsilon$. Thus letting $t\to 0^+$ gives

$$\lim_{t\to 0^+}\int_{\mathbb{R}^N} v_j\mathrm{d}F_t(v)=\int_{\mathbb{R}^N} v_j\chi(\varepsilon v)\mathrm{d}F_0(v)+O(\varepsilon).$$

Then letting $\varepsilon\to 0^+$ leads to the first equality in (4.5) for $j=1,2,\dots,N$. Next using $|v|^2\ge|v|^2\chi(\varepsilon v)$ and the inequality (4.4) we have

$$\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_0\ge\lim_{t\to 0^+}\int_{\mathbb{R}^N}|v|^2\mathrm{d}F_t\ge\lim_{t\to 0^+}\int_{\mathbb{R}^N}|v|^2\chi(\varepsilon v)\mathrm{d}F_t(v)=\int_{\mathbb{R}^N}|v|^2\chi(\varepsilon v)\mathrm{d}F_0(v)$$

which leads to the second equality in (4.5) by letting $\varepsilon\to 0^+$.

Next let's prove (4.2) and (4.3). Given any $s\ge 0$ and $\varphi\in L^\infty_{-s}\cap C^2(\mathbb{R}^N)$. For any $0<\delta<T<\infty$, by denoting

$$C_{\delta,T,s}=\sup_{\delta\le t\le T}\|F_t\|_s^2<\infty$$

and using (1.11) we have

$$\left|\int_{\mathbb{R}^N}\varphi\mathrm{d}F_{t_1}-\int_{\mathbb{R}^N}\varphi\mathrm{d}F_{t_2}\right|\le C_\varphi A_2 C_{\delta,T,s}|t_1-t_2| \qquad \forall\, t_1,t_2\in[\delta,T]\,.$$

So

$$t \mapsto \int_{\mathbb{R}^N} \varphi \mathrm{d}F_t \quad \text{is continuous in } t \in (0, \infty)\,. \tag{4.6}$$

In order to prove (4.2), we need only to show that for any fixed $t > 0$ and any sequence $\{t_n\} \subset [t/2,\, 3t/2]$ satisfying $t_n \to t$ $(n \to \infty)$ we have

$$\lim_{n\to\infty} \langle Q(F_{t_n}, F_{t_n}), \varphi\rangle = \langle Q(F_t, F_t), \varphi\rangle\ . \tag{4.7}$$

This is an application of Proposition 2.2. In fact by Proposition 2.1 we know that $(v, v_*) \mapsto L_B[\Delta\varphi](v, v_*)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$, and as shown above

$$\frac{|L_B\,[\Delta\varphi]\,(v, v_*)|}{\langle v\rangle^{s_1} + \langle v_*\rangle^{s_1}} \le C_\varphi A_2 \frac{(\langle v\rangle^s + \langle v_*\rangle^s)|v - v_*|^{2+\gamma}}{\langle v\rangle^{s_1} + \langle v_*\rangle^{s_1}} \to 0$$

for all $s_1 > s + 2 + \gamma$ as $|v|^2 + |v_*|^2 \to \infty$. Since

$$\sup_{t/2 \le \tau \le 3t/2} \|F_\tau\|_{s_1} < \infty,$$

it follows from Proposition 2.2 and the weak-star convergence $F_{t_n} \rightharpoonup F_t$ $(n \to \infty)$ (see (4.6)) that (4.7) and therefore (4.2) hold true.

The differential equation (4.3) follows from the continuity property (4.2) and from the equation (4.1) which has been proven to hold for all $\varphi \in L^\infty_{-s} \cap C^2(\mathbb{R}^N)$.

Now for any $s \ge 6$, applying (4.3) to $\varphi(v) = \langle v\rangle^s$, which belongs to $L^\infty_{-s} \cap C^2(\mathbb{R}^N)$, and applying Lemma 3.7 with $p = s/2$ we have for any $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F_t\|_s = \langle Q(F_t, F_t),\, \langle\cdot\rangle^s\rangle \le 2^{s+1} A_2 \|F_0\|_2 \|F_t\|_s - \frac{1}{4} A_2 \|F_0\|_0 \|F_t\|_{s+\gamma}.$$

Since, by using the inequality (3.29),

$$\|F_t\|_{s+\gamma} \ge (\|F_0\|_2)^{-\frac{\gamma}{s-2}} \left(\|F_t\|_s\right)^{1+\frac{\gamma}{s-2}}$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F_t\|_s \le 2^{s+1} A_2 \|F_0\|_2 \|F_t\|_s - \frac{1}{4} A_2 \|F_0\|_0 (\|F_0\|_2)^{-\frac{\gamma}{s-2}} \left(\|F_t\|_s\right)^{1+\frac{\gamma}{s-2}} \quad \forall\, t > 0\,.$$

Thus using Lemma 3.8 we obtain

$$\|F_t\|_s \le \left(\frac{2^{s+1} A_2 \|F_0\|_2}{\frac{1}{4} A_2 \|F_0\|_0 (\|F_0\|_2)^{-\frac{\gamma}{s-2}} \left(1 - \exp(-\frac{\gamma}{s-2} 2^{s+1} A_2 \|F_0\|_2 t)\right)} \right)^{\frac{s-2}{\gamma}} \quad \forall\, t > 0\,.$$

Since $s \ge 6$ implies $2^s \ge 8(s-2)$, this gives

$$\frac{\gamma}{s-2} 2^{s+1} A_2 \|F_0\|_2 \ge 16 A_2 \|F_0\|_2 \gamma =: \beta$$

and hence

$$\|F_t\|_s \le \|F_0\|_2 \left(\frac{\|F_0\|_2}{\|F_0\|_0} \cdot \frac{2^{s+3}}{1 - e^{-\beta t}} \right)^{\frac{s-2}{\gamma}}\ , \quad t > 0\,, \quad s \ge 6\,.$$

Applying this estimate to $s = 6$ we also obtain that for any $2 \le s < 6$

$$\|F_t\|_s \le (\|F_0\|_2)^{\frac{6-s}{4}} (\|F_t\|_6)^{\frac{s-2}{4}} \le (\|F_0\|_2)^{\frac{6-s}{4}} (\|F_0\|_2)^{\frac{s-2}{4}} \left(\frac{\|F_0\|_2}{\|F_0\|_0} \cdot \frac{2^9}{1-e^{-\beta t}} \right)^{\frac{4}{\gamma} \times \frac{s-2}{4}}$$

$$= \|F_0\|_2 \left(\frac{\|F_0\|_2}{\|F_0\|_0} \cdot \frac{2^9}{1-e^{-\beta t}} \right)^{\frac{s-2}{\gamma}} .$$

Maximizing the two cases gives $\max\{2^{s+3}, 2^9\} \le 2^{s+7}$ for all $s \ge 2$ and thus

$$\|F_t\|_s \le \|F_0\|_2 \left(\frac{\|F_0\|_2}{\|F_0\|_0} \cdot \frac{2^{s+7}}{1-e^{-\beta t}} \right)^{\frac{s-2}{\gamma}} \qquad \forall\, t > 0\,, \quad \forall\, s \ge 2. \tag{4.8}$$

The estimate (1.24) now follows from (4.8) since by using the inequality

$$\frac{1}{1-e^{-\beta t}} \le \left(1 + \frac{1}{\beta}\right) \left(1 + \frac{1}{t}\right)$$

we have

$$\|F_t\|_s \le \|F_0\|_2 \left\{ 2^{s+7} \frac{\|F_0\|_2}{\|F_0\|_0} \left(1 + \frac{1}{\beta}\right) \right\}^{\frac{s-2}{\gamma}} \left(1 + \frac{1}{t}\right)^{\frac{s-2}{\gamma}} = \mathcal{K}_s(F_0) \left(1 + \frac{1}{t}\right)^{\frac{s-2}{\gamma}} .$$

Note that from (4.8) and $0 < \gamma \le 2$ we also have

$$\|F_t\|_s \le \frac{\|F_0\|_0}{2^{s+7}} \left(\frac{\|F_0\|_2}{\|F_0\|_0} \cdot \frac{2^{s+7}}{(1-e^{-\beta t})} \right)^{\frac{s}{\gamma}} \qquad \forall\, t > 0\,, \quad \forall\, s \ge 2 \tag{4.9}$$

which will be used below.

Now we are going to prove the exponential moment production estimate (1.26). Let p, q be defined through the following relation (as used in Lemma 3.7)

$$q = ap \quad \text{with} \quad a = \frac{2}{\gamma} .$$

Also recall that F_t conserves the mass and energy, i.e. $\|F_t\|_0 = \|F_0\|_0$, $\|F_t\|_2 = \|F_0\|_2$. We consider two cases:

Case 1. $0 < \gamma < 2$. In this case we have $a > 1$. By Lemma 3.7 we have for all $t > 0$ and $q \ge 3a$ (i.e. for all $p \ge 3$)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Z_q(t) &= \frac{\left\langle Q(F_t, F_t),\, \langle \cdot \rangle^{2p} \right\rangle}{\Gamma(q) \|F_0\|_0} \\ &\le \left(C_a q^{2-a} + C_a q^{3-a} \varepsilon_p \right) A_2 \|F_0\|_0 Z_p^*(t) + \frac{1}{2} A_2 \|F_0\|_2 Z_q(t) - \frac{q}{16} A_2 \|F_0\|_0 Z_q(t)^{1+\frac{1}{q}} , \end{aligned}$$

where

$$Z_q(t) = \frac{\|F_t\|_{\gamma q}}{\Gamma(q) \|F_0\|_0} , \quad Z_p^*(t) = \max_{k \in \{1,2,\dots,k_p\}} \{ Z_{ak+1}(t) Z_{a(p-k)}(t)\,,\, Z_{ak}(t) Z_{a(p-k)+1}(t) \} .$$

Using $a = 2/\gamma > 1$ and Lemma 3.4 we have

$$C_a q^{2-a} + C_a q^{3-a} \varepsilon_p = o(1) q \quad (q \to \infty)$$

so that there is a positive integer n_0, depending only on $b(\cdot)$ and γ, such that

$$n_0 \delta \ge 3a \quad \text{and} \quad C_a q^{2-a} + C_a q^{3-a} \varepsilon_p \le \frac{q}{32} \qquad \forall\, q \ge n_0 \delta, \quad \text{where} \quad \delta = a - 1 .$$

Since

$$q \ge n_0\delta \implies \frac{1}{2}A_2\|F_0\|_2 < 16A_2\|F_0\|_2\gamma q = \beta q,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_q(t) \le \frac{A_2\|F_0\|_0 q}{32}Z_p^*(t) + \beta q Z_q(t) - \frac{q}{16}A_2\|F_0\|_0 Z_q(t)^{1+\frac{1}{q}} \qquad \forall\, q \ge n_0\delta\,. \tag{4.10}$$

Let

$$\Theta = 2^{\gamma n_0\delta+7}\frac{\|F_0\|_2}{\|F_0\|_0}\,, \quad Y_q(t) = \left(\frac{\Theta}{1-e^{-\beta t}}\right)^q\,, \quad t>0\,.$$

Then Y_q satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_q(t) = \beta q Y_q(t) - \frac{\beta q}{\Theta}(Y_q(t))^{1+\frac{1}{q}}\,, \quad t>0\,; \quad Y_q(0+) = \infty\,.$$

We now prove that

$$Z_q(t) \le Y_q(t) \qquad \forall\, t>0\,, \quad \forall\, q \ge 1\,. \tag{4.11}$$

To do this, it suffices to show that

$$Z_q(t) \le Y_q(t) \qquad \forall\, t>0\,, \quad \forall\, q\in[1,\, n\delta]\,, \quad n = n_0, n_0+1, n_0+2, \dots\,. \tag{4.12}$$

First of all it is easily seen that (4.12) holds for $n=n_0$. In fact by definitions of $Z_q(t)$ and $Y_q(t)$ and using the inequality $\Gamma(q) > 1/2$ ($\forall\, q\ge 1$) and (4.9) we have for all $1\le q\le n_0\delta$

$$Z_q(t) \le 2\frac{\|F_t\|_{\gamma q}}{\|F_0\|_0} \le \left(\frac{\|F_0\|_2}{\|F_0\|_0}\cdot\frac{2^{\gamma q+7}}{1-e^{-\beta t}}\right)^q \le Y_q(t) \quad \forall\, t>0.$$

Suppose that (4.12) holds for an integer $n\ge n_0$. Take any $q\in[n\delta\,,\,(n+1)\delta]$. Then $q\ge n\delta\ge n_0\delta$ and so (4.10) holds for such q. Recall that $ap=q$. Since for all integer $1\le k\le k_p = [(p+1)/2]$ there hold

$$\begin{cases} 1 < ak < ak+1 \le \frac{(n+1)\delta+a}{2}+1 < n\delta, \\ 1 < a(p-k) < a(p-k)+1 \le q-\delta \le n\delta \end{cases}$$

it follows from the inductive hypothesis that

$$\begin{cases} Z_{ak+1}(t)Z_{a(p-k)}(t) \le Y_{ak+1}(t)Y_{a(p-k)}(t) = Y_{q+1}(t)\,, \\ Z_{ak}(t)Z_{a(p-k)+1}(t) \le Y_{ak}(t)Y_{a(p-k)+1}(t) = Y_{q+1}(t)\,. \end{cases}$$

Therefore by definitions of $Z_p^*(t), Y_q(t)$ we obtain

$$Z_p^*(t) \le Y_{q+1}(t) = Y_q(t)^{1+\frac{1}{q}}\,, \quad \forall\, t>0\,, \quad \forall\, q\in[n\delta,\,(n+1)\delta]$$

and hence by (4.10)

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_q(t) \le \beta q Z_q(t) + \frac{A_2\|F_0\|_0}{32}qY_q(t)^{1+\frac{1}{q}} - \frac{A_2\|F_0\|_0}{16}qZ_q(t)^{1+\frac{1}{q}} \qquad \forall\, t>0$$

for all $q\in[n\delta,\,(n+1)\delta]$. From this we obtain the following inequality:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}Z_q(t)\right)1_{\{Z_q(t)>Y_q(t)\}} \le \left(\beta q Z_q(t) - \frac{\beta q}{\Theta}Z_q(t)^{1+\frac{1}{q}}\right)1_{\{Z_q(t)>Y_q(t)\}} \quad \forall\, t>0$$

where we used the obvious fact that

$$\frac{A_2\|F_0\|_0}{32} > \frac{\beta}{\Theta}.$$

Thus applying Lemma 3.8 we conclude $Z_q(t) \le Y_q(t)$ for all $t > 0$. This together with the inductive hypotheses implies that $Z_q(t) \le Y_q(t)$ for all $t > 0$ and all $q \in [1\,,\,(n+1)\delta]$. This proves (4.12) and thus (4.11) holds true.

Now let

$$\alpha(t) = \frac{1-e^{-\beta t}}{2\Theta}\,, \quad t > 0\,. \tag{4.13}$$

Then by definitions of $Z_q(t), Y_q(t)$ and $Z_q(t) \le Y_q(t)$ we have for all $t > 0$

$$\frac{(\alpha(t))^q \|F_t\|_{\gamma q}}{q!\|F_0\|_0} \le (\alpha(t))^q Z_q(t) \le (\alpha(t))^q Y_q(t) = \frac{1}{2^q}, \quad q = 1, 2, \dots$$

and thus

$$\int_{\mathbb{R}^N} e^{\alpha(t)\langle v\rangle^\gamma} \mathrm{d}F_t(v) = \|F_0\|_0 + \sum_{q=1}^{\infty} \frac{(\alpha(t))^q}{q!}\|F_t\|_{\gamma q} \le 2\|F_0\|_0\,.$$

Case 2. $\gamma = 2$. In this case we have $a = 1$ hence $q = p$. From part (II) of Lemma 3.7 with p_1, q_1 and η given in (3.13)-(3.14), we have for all $p \ge (12A^*_{p_1}/A_0)^{2q_1}$ (which is larger than 5)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Z_p(t) &\le 48A^*_{p_1}p^{1-\eta}(\log p)\|F_0\|_0 \widetilde{Z}^*_p(t)\\ &\quad + \left(12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4}\right)\|F_0\|_2 Z_p(t) - \frac{A_0\|F_0\|_0}{16}pZ_p(t)^{1+\frac{1}{p}}\end{aligned}$$

where

$$Z_p(t) = \frac{\|F_t\|_{2p}}{\Gamma(p)\|F_0\|_0}\,, \quad \widetilde{Z}^*_p(t) = \max_{k\in\{1,2,\dots,k_p\}} Z_{k+1}(t)Z_{p-k}(t)\,, \quad t > 0\,.$$

Let us fix an integer $n_0 \ge (12A^*_{p_1}/A_0)^{2q_1}$ such that

$$48A^*_{p_1}p^{1-\eta}\log p \le \frac{A_2}{32}p\,, \quad 12A^*_{p_1}p^{1-\eta} + \frac{A_0}{4} \le 32A_2 p \qquad \forall\, p \ge n_0\,.$$

Recalling $\beta = 32A_2\|F_0\|_2$ for $\gamma = 2$, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_p(t) \le \frac{A_2\|F_0\|_0}{32}p\widetilde{Z}^*_p(t) + \beta p Z_p(t) - \frac{A_2\|F_0\|_0}{16}pZ_p(t)^{1+\frac{1}{p}} \qquad \forall\, p \ge n_0\,. \tag{4.14}$$

It will be clear that in the present case all p can be chosen integers. Let

$$\Theta = 2^{2n_0+7}\frac{\|F_0\|_2}{\|F_0\|_0}\,, \quad Y_p(t) = \left(\frac{\Theta}{1-e^{-\beta t}}\right)^p\,, \quad t > 0\,;\ p \ge 1\,.$$

Then Y_p satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_p(t) = \beta p Y_p(t) - \frac{\beta p}{\Theta}Y_p(t)^{1+\frac{1}{p}}\,, \quad t > 0\,; \quad Y_p(0+) = \infty\,.$$

We now prove that

$$Z_p(t) \le Y_p(t) \qquad \forall\, t > 0\,, \quad p = 1, 2, 3, \dots \tag{4.15}$$

As shown in the Case 1 one sees that (4.15) holds for all integer $1 \le p \le n_0$. Suppose that (4.15) holds true for some integer $p-1 \ge n_0$. Let us check the case p. By $p-1 \ge n_0 > 5$

we have $k_p+1 \le (p+1)/2+1 \le p-1$ and so $Z_{k+1}(t)Z_{p-k}(t) \le Y_{k+1}(t)Y_{p-k}(t) = (Y_p(t))^{1+\frac{1}{p}}$ hold for all $k \in \{1, 2, ..., k_p\}$. So

$$\widetilde{Z}_p^*(t) = \max_{k\in\{1,2,\dots,k_p\}} Z_{k+1}(t)Z_{p-k}(t) \le Y_p(t)^{1+\frac{1}{p}}$$

hence from (4.14) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_p(t) \le \beta p Z_p(t) + \frac{A_2\|F_0\|_0}{32}pY_p(t)^{1+\frac{1}{p}} - \frac{A_2\|F_0\|_0}{16}pZ_p(t)^{1+\frac{1}{p}} \qquad \forall\, t>0$$

which together with $\frac{A_2\|F_0\|_0}{32} > \frac{\beta}{\Theta}$ implies the inequality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}Z_p(t)\right)1_{\{Z_p(t)>Y_p(t)\}} \le \left(\beta p Z_p(t) - \frac{\beta p}{\Theta}Z_p(t)^{1+\frac{1}{p}}\right)1_{\{Z_p(t)>Y_p(t)\}} \quad \forall\, t>0.$$

Applying Lemma 3.8 we then conclude that $Z_p(t) \le Y_p(t)\ \forall\, t>0$. This proves (4.15).

As shown above we obtain with the function $\alpha(t)$ defined in (4.13) that

$$\int_{\mathbb{R}^N} e^{\alpha(t)\langle v\rangle^2}\mathrm{d}F_t(v) \le 2\|F_0\|_0 \qquad \forall\, t>0\,.$$

This completes Step 1.

Step 2. Construction of solutions for absolutely continuous measures. Suppose that F_0 is absolutely continuous with respect to the Lebesgue measure, i.e. $\mathrm{d}F_0(v) = f_0(v)\mathrm{d}v$, and suppose that (moment bounds and finite entropy)

$$0 \le f_0 \in \bigcap_{s\ge 0} L^1_s(\mathbb{R}^N) \quad \text{ and } \quad 0 < \int_{\mathbb{R}^N} f_0(v)|\log f_0(v)|\mathrm{d}v < \infty\,.$$

In this case we prove that there exists $\{f_t\}_{t\ge0} \subset \bigcap_{s\ge0} L^1_s(\mathbb{R}^N)$ such that the measure F_t defined by $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v$ is a conservative measure weak solution of Eq. (1.1) associated with the initial datum F_0 and F_t satisfies the moment production estimates (1.24) and (1.26).

To do this we consider some bounded truncations B_n of the kernel B:

$$B_n(z,\sigma) = \min\{|z|^\gamma,\, n\}\min\{b(\cos\theta),\, n\}\,, \quad n = 1, 2, \dots$$

It is well known that for every $n \ge 1$ the Eq. (1.1) with the bounded kernel B_n has a unique conservative solution $f^n_t(v)$ satisfying $f^n_0(v) = f_0(v)$ and $f^n \in C^1([0,\infty); L^1_s(\mathbb{R}^N)) \cap L^\infty_{\mathrm{loc}}([0,\infty); L^1_s(\mathbb{R}^N))$ for all $s \ge 0$, and

$$\sup_{n\ge1,\,t\ge0}\int_{\mathbb{R}^N} f^n_t(v)\left(1+|v|^2+|\log f^n_t(v)|\right)\mathrm{d}v < \infty\,. \tag{4.16}$$

Let $Q_{B_n}(\cdot,\cdot)$ (collision operator) and $A_{n,2}$ (angular momentum defined in **(H0)**) correspond to the kernel B_n, and define $\mathrm{d}F^n_t(v) = f^n_t(v)\mathrm{d}v$. Then $\|F^n_t\|_2 = \|F^n_0\|_2 = \|F_0\|_2$ and from the proof of Lemmas 3.6-3.7 we see that by omitting the negative term in the proofs of the two lemmas and noting that $A_{n,2} \le A_2$ we have for all $p \ge 3$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F^n_t\|_{2p} = \left\langle Q_{B_n}(F^n_t, F^n_t), \langle\cdot\rangle^{2p}\right\rangle \le 2^{2p+1}A_2\|F_0\|_2\|F^n_t\|_{2p}\,.$$

Thus for all $s \ge 6$, letting $p = s/2$ and recalling $\|f^n_t\|_{L^1_s} = \|F^n_t\|_s$ we obtain

$$\sup_{n\ge1}\|f^n_t\|_{L^1_s} \le \|f_0\|_{L^1_s}\exp\left(2^{s+1}A_2\|F_0\|_2 t\right) \qquad \forall\, t\ge 0\,.$$

From this and the basic estimate (1.11) we get for any $\varphi \in C_b^2(\mathbb{R}^N)$ and any $T \in (0,\infty)$

$$\Big|\int_{\mathbb{R}^N} \varphi(v) f^n_{t_1}(v)\mathrm{d}v - \int_{\mathbb{R}^N} \varphi(v) f^n_{t_2}(v)\mathrm{d}v\Big| \le C_{\varphi,T}|t_1-t_2| \qquad \forall\, t_1,t_2\in[0,T]\,.$$

This together with (4.16) implies for any $\psi \in L^\infty(\mathbb{R}^N)$ and any $T\in(0,\infty)$

$$\sup_{t_1,t_2\in[0,T],\,|t_1-t_2|\le\delta;\,n\ge1}\left|\int_{\mathbb{R}^N}\psi f^n_{t_1}\mathrm{d}v - \int_{\mathbb{R}^N}\psi f^n_{t_2}\mathrm{d}v\right| \to 0 \quad \text{as} \quad \delta\to 0^+\,. \tag{4.17}$$

Since (4.16) implies that for every $t\ge 0$, $\{f^n_t\}_{n=1}^\infty$ is L^1-weakly relatively compact, it follows from diagonal argument and (4.17) that there is a subsequence of $\{n\}$ (independent of t), still denoted as $\{n\}$, and a nonnegative measurable function $(t,v)\mapsto f_t(v)$ on $[0,\infty)\times\mathbb{R}^N$ satisfying $f_t\in L^1(\mathbb{R}^N)$ $(\forall\, t\ge0)$ such that for all $\psi\in L^\infty(\mathbb{R}^N)$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\psi f^n_t\mathrm{d}v = \int_{\mathbb{R}^N}\psi f_t\mathrm{d}v \quad \forall\, t\ge 0\,. \tag{4.18}$$

And consequently

$$f_t \in \bigcap_{s\ge0} L^1_s(\mathbb{R}^N) \quad \forall\, t\ge0\,,$$

and

$$\sup_{t\ge0}\|f_t\|_{L^1_2}\le\|f_0\|_{L^1_2}\,,\quad \sup_{0\le t\le T}\|f_t\|_{L^1_s}<\infty \quad \forall\, 0<T<\infty\,,\quad \forall\, s\ge0\,, \tag{4.19}$$

and for any $s>0$ and any $\psi\in L^\infty(\mathbb{R}^N)$

$$t\mapsto\int_{\mathbb{R}^N}\psi f_t\mathrm{d}v \quad \text{is continuous on} \quad [0,\infty)\,. \tag{4.20}$$

Now we are going to show that f_t (or equivalently the measure F_t defined by $\mathrm{d}F_t(v)=f_t(v)\mathrm{d}v$) is a conservative weak solution of Eq. (1.1) with the kernel B. Given any $\varphi\in C_b^2(\mathbb{R}^N)$, we have by (1.11) and $B_n\le B$

$$\sup_{n\ge1}\frac{|L_{B_n}[\Delta\varphi](v,v_*)|}{\langle v\rangle^s+\langle v_*\rangle^s}\le A_2C_\varphi\frac{|v-v_*|^{2+\gamma}}{\langle v\rangle^s+\langle v_*\rangle^s}\to0 \quad (|v|^2+|v_*|^2\to\infty)$$

for $s>2+\gamma$. Moreover by Proposition 2.1, $L_{B_n}[\Delta\varphi](v,v_*)$, $L_B[\Delta\varphi](v,v_*)$ are all continuous on $(v,v_*)\in\mathbb{R}^N\times\mathbb{R}^N$, and

$$\lim_{n\to\infty}\sup_{|v|+|v_*|\le R}|L_{B_n}[\Delta\varphi](v,v_*)-L_B[\Delta\varphi](v,v_*)|=0 \qquad \forall\, 0<R<\infty\,.$$

It follows from (4.18) and Proposition 2.2 that

$$\sup_{0\le t\le T}\iint_{\mathbb{R}^N\times\mathbb{R}^N}|L_B[\Delta\varphi](v,v_*)|\,f_t(v)f_t(v_*)\mathrm{d}v\mathrm{d}v_*<\infty \qquad \forall\, 0<T<\infty\,,$$

$$\langle Q_{B_n}(f^n_t,f^n_t),\varphi\rangle\to\langle Q_B(f_t,f_t),\varphi\rangle \quad (n\to\infty) \quad \forall\, t\ge0\,.$$

Again using Proposition 2.2 and (4.20) we conclude that

$$t\mapsto\langle Q_B(f_t,f_t),\varphi\rangle \quad \text{is continuous on} \quad [0,\infty)\,.$$

Finally using the dominated convergence theorem (in the t variable) we conclude that

$$\int_{\mathbb{R}^N}\varphi f_t\mathrm{d}v=\int_{\mathbb{R}^N}\varphi f_0\mathrm{d}v+\int_0^t\langle Q_B(f_\tau,f_\tau),\varphi\rangle\,\mathrm{d}\tau \qquad \forall\, t\ge0\,.$$

Thus f_t is a weak solution of Eq. (1.1). Let F_t be defined by $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v$. Then from $\|F_t\|_s = \|f_t\|_{L^1_s}$, (4.19), and Step 1 we conclude that F_t is a conservative measure weak solution of Eq. (1.1) associated with the initial datum F_0 and satisfies the moment production estimates (1.24) and (1.26).

Step 3. The approximation argument and conclusion. Let F_0 be the given measure in $\mathcal{B}_2^+(\mathbb{R}^N)$ with $\|F_0\|_0 \neq 0$. We shall prove the existence of a measure weak solution F_t that has all properties listed in the theorem.

First if $F_0 = c\delta_{v=v_0}$ $(c > 0)$ is a Dirac mass, then it is easily checked that the measure $F_t \equiv c\delta_{v=v_0}$ is a measure weak solution of Eq.(1.1) and apparently it conserves the mass, momentum and energy and has finite moments of all orders. By Step 1 we conclude that F_t satisfies the moment production estimates (1.24)-(1.26).

Suppose F_0 is not a Dirac mass. We shall use **Mehler transform**: Let

$$\rho = \|F_0\|_0\,, \quad v_0 = \frac{1}{\rho}\int_{\mathbb{R}^N} v\mathrm{d}F_0(v)\,, \quad T = \frac{1}{N\rho}\int_{\mathbb{R}^N} |v - v_0|^2\mathrm{d}F_0(v)\,. \tag{4.21}$$

Then $T > 0$ so that the Maxwellian used in the Mehler transform can be defined:

$$M(v) = \frac{e^{-|v|^2/2T}}{(2\pi T)^{N/2}}\,, \qquad v \in \mathbb{R}^N\,. \tag{4.22}$$

The Mehler transform of F_0 is defined by

$$f_0^n(v) = e^{Nn}\int_{\mathbb{R}^N} M\Big(e^n\Big(v - v_0 - \sqrt{1 - e^{-2n}}\,(v_* - v_0)\Big)\Big)\,\mathrm{d}F_0(v_*)\,, \quad n \geq 1\,. \tag{4.23}$$

It is well known that

$$\int_{\mathbb{R}^N}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} f_0^n(v)\mathrm{d}v = \int_{\mathbb{R}^N}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}F_0(v)$$

and for all $\psi \in L^\infty_{-2} \cap C(\mathbb{R}^N)$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\psi(v)f_0^n(v)\mathrm{d}v = \int_{\mathbb{R}^N}\psi(v)\mathrm{d}F_0(v)\,.$$

For every n, choose $K_n > n$ such that

$$\int_{\mathbb{R}^N}\Big(f_0^n(v) - \min\{f_0^n(v),\, K_n\}e^{-\frac{|v|^2}{K_n}}\Big)\langle v\rangle^2\mathrm{d}v \leq \frac{\|F_0\|_0}{2n}\,. \tag{4.24}$$

Then let

$$\widetilde{f_0^n}(v) = \min\{f_0^n(v),\, K_n\}e^{-|v|^2/n}, \qquad \mathrm{d}F_0^n(v) = \widetilde{f_0^n}(v)\mathrm{d}v\,.$$

We need to prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\psi\mathrm{d}F_0^n = \int_{\mathbb{R}^N}\psi\mathrm{d}F_0 \qquad \forall\,\psi \in L^\infty_{-2}C(\mathbb{R}^N)\,. \tag{4.25}$$

Indeed we have

$$\Big|\int_{\mathbb{R}^N}\psi\mathrm{d}F_0^n - \int_{\mathbb{R}^N}\psi\mathrm{d}F_0\Big| \leq \Big|\int_{\mathbb{R}^N}\psi(\widetilde{f_0^n} - f_0^n)\mathrm{d}v\Big| + \Big|\int_{\mathbb{R}^N}\psi f_0^n\mathrm{d}v - \int_{\mathbb{R}^N}\psi\mathrm{d}F_0\Big|.$$

The second term converges to zero ($n \to \infty$). The first term also goes to zero: By (4.24) we have

$$\Big| \int_{\mathbb{R}^N} \psi(\widetilde{f}_0^n - f_0^n)\mathrm{d}v\Big| \le C \int_{\mathbb{R}^N} \langle v\rangle^2 |\widetilde{f}_0^n - f_0^n|\mathrm{d}v \le \frac{C}{2n}\,.$$

Since for every n, $\widetilde{f}_0^n$ satisfies the condition in the Step 2, there is a conservative measure weak solution F_t^n of Eq. (1.1) with the kernel B and the initial data F_0^n, such that F_t^n satisfies the moment estimates

$$\|F_t^n\|_s \le \mathcal{K}_s(F_0^n)(1+1/t)^{\frac{s-2}{\gamma}} \qquad \forall\, t>0\,, \quad \forall\, s\ge 2.$$

Here recall that $\mathcal{K}_s(\cdot)$ is defined in (1.25). By the convergence (4.25) we have

$$\lim_{n\to\infty} \mathcal{K}_s(F_0^n) = \mathcal{K}_s(F_0) \qquad \forall\, s \ge 2\,.$$

Thus for any $s \ge 2$, $C_s^* := \sup_{n\ge 1} \mathcal{K}_s(F_0^n) < \infty$ and hence

$$\sup_{n\ge1} \|F_t^n\|_s \le C_s^*\,(1+1/t)^{\frac{s-2}{\gamma}} \qquad \forall\, t>0\,, \quad \forall\, s \ge 2\,. \tag{4.26}$$

Next we prove the equi-continuity of $\{F_t^n\}_{n=1}^\infty$ in $t\in[0,\infty)$ (in particular in the neighborhood of $t=0$). It is only in this part that the logarithm $|\log(\sin\theta)|$ comes into play. Let

$$\lambda(\theta) := \frac{1}{1+|\log(\sin\theta)|}\,, \quad 0<\theta<\pi\,.$$

By (1.11) and $0<\gamma\lambda(\theta)\le\gamma\le 2$ we have for any $\varphi\in C_b^2(\mathbb{R}^N)$

$$\left|\int_{\mathbb{S}^{N-2}(\mathbf{n})} \Delta\varphi\,\mathrm{d}\omega\right| \le C_\varphi \left|\int_{\mathbb{S}^{N-2}(\mathbf{n})} \Delta\varphi\,\mathrm{d}\omega\right|^{\frac{2-\gamma\lambda(\theta)}{2}} \le C_\varphi |v-v_*|^{2-\gamma\lambda(\theta)}(\sin\theta)^{2-\gamma\lambda(\theta)}$$

where here and below C_φ only depends on φ and N. Then by using

$$|v-v_*|^{\gamma+2-\gamma\lambda(\theta)} \le 8\left(\langle v\rangle^{\gamma+2-\gamma\lambda(\theta)} + \langle v_*\rangle^{\gamma+2-\gamma\lambda(\theta)}\right)$$

and $(\sin\theta)^{-\gamma\lambda(\theta)} = e^{\gamma(1-\lambda(\theta))} \le e^2$ and recalling (1.9) we obtain

$$|L_B\,[\Delta\varphi]\,(v,v_*)| \le C_\varphi \int_0^\pi b(\cos\theta)\sin^N\theta\left(\langle v\rangle^{\gamma+2-\gamma\lambda(\theta)} + \langle v_*\rangle^{\gamma+2-\gamma\lambda(\theta)}\right)\mathrm{d}\theta\,.$$

So for all $t>0$ (using Fubini's theorem and (4.26))

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N} |L_B\,[\Delta\varphi]\,(v,v_*)|\,\mathrm{d}F_t^n(v)\mathrm{d}F_t^n(v_*) \\ &\le C_\varphi \|F_0\|_0 \int_0^\pi b(\cos\theta)\sin^N\theta\,\|F_t^n\|_{\gamma+2-\gamma\lambda(\theta)}\mathrm{d}\theta \\ &\le C_{\varphi,F_0}\int_0^\pi b(\cos\theta)\sin^N\theta\,\left(1+\frac{1}{t}\right)^{1-\lambda(\theta)}\,\mathrm{d}\theta\,. \end{aligned} \tag{4.27}$$

Thus for all $t_1, t_2 \in [0, \infty)$ we compute (assuming $t_1 < t_2$)

$$\begin{aligned}
(4.28) \qquad & \int_{t_1}^{t_2} \mathrm{d}t \iint_{\mathbb{R}^N \times \mathbb{R}^N} |L_B[\Delta\varphi](v, v_*)| \,\mathrm{d}F_t^n(v)\mathrm{d}F_t^n(v_*) \\
& \le C_{\varphi,F_0} \int_0^\pi b(\cos\theta)\sin^N\theta \,\mathrm{d}\theta\,(1 + t_2 - t_1)^{1-\lambda(\theta)} \int_0^{t_2 - t_1} t^{\lambda(\theta)-1}\mathrm{d}t \\
& = C_{\varphi,F_0} \int_0^\pi b(\cos\theta)\sin^N\theta\,(1 + |\log(\sin\theta)|)\,(1 + t_2 - t_1)^{1-\lambda(\theta)}\,(t_2 - t_1)^{\lambda(\theta)}\mathrm{d}\theta \\
& =: C_{\varphi,F_0}\Omega(t_2 - t_1).
\end{aligned}$$

Since

$$|\langle Q(F_t^n, F_t^n), \varphi\rangle| \le \iint_{\mathbb{R}^N \times \mathbb{R}^N} |L_B[\Delta\varphi](v, v_*)| \,\mathrm{d}F_t^n(v)\mathrm{d}F_t^n(v_*)\,,$$

it follows that

$$\begin{aligned}
\sup_{n\ge1}\Big|\int_{\mathbb{R}^N}\varphi \mathrm{d}F_{t_2}^n - \int_{\mathbb{R}^N}\varphi \mathrm{d}F_{t_1}^n\Big| &\le \sup_{n\ge1}\Big|\int_{t_1}^{t_2}|\langle Q(F_t^n, F_t^n), \varphi\rangle|\mathrm{d}t\Big| \\
&\le C_{\varphi,F_0}\Omega(|t_2 - t_1|) \to 0
\end{aligned}$$

as $|t_1 - t_2| \to 0$. We then deduce for any $\psi \in C_c(\mathbb{R}^N)$ that

$$(4.29) \qquad \Lambda_\psi(\delta) := \sup_{|t_1-t_2|\le\delta;\, n\ge1}\left|\int_{\mathbb{R}^N}\psi \mathrm{d}F_{t_1}^n - \int_{\mathbb{R}^N}\psi \mathrm{d}F_{t_2}^n\right| \to 0 \quad \text{as} \quad \delta \to 0^+\,.$$

Since $C_c(\mathbb{R}^N)$ is separated, it follows from a diagonal argument that there is a subsequence of $\{n\}$ (independent of t), still denoted by $\{n\}$, and a family $\{F_t\}_{t\ge0} \subset \mathcal{B}_2^+(\mathbb{R}^N)$, such that

$$(4.30) \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N}\psi \mathrm{d}F_t^n = \int_{\mathbb{R}^N}\psi \mathrm{d}F_t \qquad \forall\, t \ge 0\,, \quad \forall\, \psi \in C_c(\mathbb{R}^N)\,.$$

Using (4.26) and the fact that F_t^n are conservative solutions we have

$$(4.31) \qquad \|F_t\|_2 \le \|F_0\|_2\,, \quad \|F_t\|_s \le C_s^*\,(1 + 1/t)^{\frac{s-2}{\gamma}} \quad \forall\, t > 0\,, \quad \forall\, s \ge 2\,.$$

Also by (4.30) and (4.29) we have

$$\Big|\int_{\mathbb{R}^N}\psi \mathrm{d}F_{t_1} - \int_{\mathbb{R}^N}\psi \mathrm{d}F_{t_2}\Big| \le \Lambda_\psi(|t_1 - t_2|)\,.$$

Hence

$$(4.32) \qquad t \mapsto \int_{\mathbb{R}^N}\psi \mathrm{d}F_t \quad \text{is continuous on} \quad [0, \infty) \quad \forall\, \psi \in C_c(\mathbb{R}^N)\,.$$

We now prove that F_t is a measure weak solution of Eq. (1.1). Given any $\varphi \in C_b^2(\mathbb{R}^N)$, by (4.31) we see that the derivation of (4.27) holds also for F_t and so

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |L_B[\Delta\varphi](v, v_*)| \,\mathrm{d}F_t(v)\mathrm{d}F_t(v_*) < \infty \quad \forall\, t > 0\,.$$

Next by Proposition 2.1 the function $(v, v_*) \mapsto L_B[\Delta\varphi](v, v_*)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ and

$$(4.33) \qquad \frac{|L_B[\Delta\varphi](v, v_*)|}{\langle v\rangle^s + \langle v_*\rangle^s} \le C_\varphi A_2 \frac{|v - v_*|^{2+\gamma}}{\langle v\rangle^s + \langle v_*\rangle^s} \to 0 \quad (|v|^2 + |v_*|^2 \to \infty)$$

for all $s > 2+\gamma$. Thus by using (4.26)-(4.30)-(4.33), Propositions 2.1 and 2.2 we have

$$\langle Q(F_t^n, F_t^n), \varphi\rangle \to \langle Q(F_t, F_t), \varphi\rangle \quad (n\to\infty) \qquad \forall\, t>0\,. \tag{4.34}$$

Similarly by using (4.31)-(4.32), Propositions 2.1 and 2.2 we conclude that

$$t \mapsto \langle Q(F_t, F_t), \varphi\rangle \quad \text{is continuous in} \quad (0,\infty)\,. \tag{4.35}$$

Note that the derivation of (4.28) also holds for F_t and hence we have for all $T\in(0,\infty)$

$$\int_0^T \mathrm{d}\tau \iint_{\mathbb{R}^N\times\mathbb{R}^N} |L_B[\Delta\varphi](v,v_*)|\, \mathrm{d}F_t(v)\mathrm{d}F_t(v_*) \le C_{\varphi,F_0}\Omega(T) < \infty\,. \tag{4.36}$$

Thus

$$t \mapsto \langle Q(F_t, F_t), \varphi\rangle \quad \text{belongs to} \quad C((0,\infty)) \cap L^1_{\mathrm{loc}}([0,\infty))\,.$$

And it also follows from (4.28)-(4.34) and the dominated convergence theorem that for all $t>0$ we have

$$\int_0^t \langle Q(F_\tau^n, F_\tau^n), \varphi\rangle \,\mathrm{d}\tau \to \int_0^t \langle Q(F_\tau, F_\tau), \varphi\rangle\,\mathrm{d}\tau \quad (n\to\infty)\,.$$

Thus in the integral equation of measures solutions F_t^n, letting $n\to\infty$ gives

$$\int_{\mathbb{R}^N} \varphi \mathrm{d}F_t = \int_{\mathbb{R}^N} \varphi\mathrm{d}F_0 + \int_0^t \langle Q(F_\tau, F_\tau), \varphi\rangle\,\mathrm{d}\tau \quad \forall\, t>0\,.$$

We have proved that F_t satisfies the conditions (i)-(ii) in the Definition 1.1 of measure weak solutions. So F_t is a measure weak solution of Eq. (1.1) associated with the initial datum F_0. Finally from the moment estimates in (4.31) and Step 1 we conclude that the solution F_t conserves mass, momentum and energy, and satisfies the moment production estimates (1.24)-(1.26). This completes the proof of Theorem 1.3.

5. Uniqueness and stability for angular cutoff: Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. We shall first prove some lemmas on how the sign decomposition of measures behaves with time integration and with the action of the collision operator.

5.1. **Sign decomposition of measures.** As usual we denote

$$\mathcal{B}(\mathbb{R}^N) = \mathcal{B}_0(\mathbb{R}^N), \quad \|\mu\| = \|\mu\|_0 = |\mu|(\mathbb{R}^N).$$

For any $\mu\in\mathcal{B}(\mathbb{R}^N)$, let μ^+, μ^- be the positive and negative parts of μ, i.e. $\mu^\pm = \frac{1}{2}(|\mu|\pm\mu)$. Let $h:\mathbb{R}^N\to\mathbb{R}$ be the Borel function satisfying $|h(v)|\equiv 1$ such that $\mathrm{d}\mu = h\mathrm{d}|\mu|$. We may call h the sign function of μ. Then $\mathrm{d}\mu^+ = \frac{1}{2}(1+h)\mathrm{d}\mu$. So for any $\mu,\nu\in\mathcal{B}(\mathbb{R}^N)$, we have

$$|\mu-\nu| = \nu-\mu+2(\mu-\nu)^+\,. \tag{5.1}$$

Let us now prove that this sign decomposition behaves well with the time integration.

Lemma 5.1 (Sign decomposition and time integration)**.** *Let $\mu_t\in C([a,\infty);\mathcal{B}(\mathbb{R}^N))$, $\nu_a\in\mathcal{B}(\mathbb{R}^N)$, and*

$$\nu_t = \nu_a + \int_a^t \mu_s \mathrm{d}s\,, \quad t\ge a,$$

and let $v\mapsto h_t(v)$ be the sign function of the measure ν_t and let $\kappa_t = (1+h_t)/2$ so that $\mathrm{d}\nu_t^+ = \kappa_t\mathrm{d}\nu_t$.

Then for any bounded Borel function ψ on $\mathbb{R}^N$, the functions

$$t \mapsto \int_{\mathbb{R}^N} \psi \mathrm{d}\mu_t\,, \quad t \mapsto \int_{\mathbb{R}^N} \psi \mathrm{d}|\mu_t| \quad \textit{and} \quad t \mapsto \int_{\mathbb{R}^N} \psi \mathrm{d}\mu_t^+$$

all belong to $L^1_{\rm loc}([a,\infty))$ and for any $t \in [a,\infty)$ we have

$$\int_{\mathbb{R}^N} \psi \mathrm{d}\nu_t = \int_{\mathbb{R}^N} \psi \mathrm{d}\nu_a + \int_a^t \mathrm{d}s \int_{\mathbb{R}^N} \psi \mathrm{d}\mu_s\,, \tag{5.2}$$

$$\int_{\mathbb{R}^N} \psi \mathrm{d}|\nu_t| = \int_{\mathbb{R}^N} \psi \mathrm{d}|\nu_a| + \int_a^t \mathrm{d}s \int_{\mathbb{R}^N} \psi h_s \mathrm{d}\mu_s\,, \tag{5.3}$$

$$\int_{\mathbb{R}^N} \psi \mathrm{d}\nu_t^+ = \int_{\mathbb{R}^N} \psi \mathrm{d}\nu_a^+ + \int_a^t \mathrm{d}s \int_{\mathbb{R}^N} \psi \kappa_s \mathrm{d}\mu_s\,. \tag{5.4}$$

Proof of Lemma 5.1. Since the half-sum of (5.2) and (5.3) is equal to (5.4), we only have to prove (5.2) and (5.3). The proof of (5.2) is easy and similar to that of (5.3). By simple function approximation, the proof of (5.3) can be reduced to the proof of that for any Borel set $E \subset \mathbb{R}^N$, $t \mapsto \int_E h_t \mathrm{d}\mu_t$ belongs to $L^1_{\rm loc}([a,\infty))$ (and so does $t \mapsto \int_{\mathbb{R}^N} \psi h_t \mathrm{d}\mu_t$ for any bounded Borel function ψ on $\mathbb{R}^N$) and

$$|\nu_t|(E) = |\nu_a|(E) + \int_a^t \mathrm{d}s \int_E h_s \mathrm{d}\mu_s\,, \quad t \in [a,\infty)\,. \tag{5.5}$$

By assumption on μ_t, the strong derivative $\frac{\mathrm{d}}{\mathrm{d}t}\nu_t = \mu_t$ exists, and

$$\|\nu_{t_1} - \nu_{t_2}\| \le \int_{t_1}^{t_2} \|\mu_s\| ds \qquad \forall\, a \le t_1 \le t_2 < \infty\,.$$

This implies that for any Borel set $E \subset \mathbb{R}^N$, $t \mapsto |\nu_t|(E)$ is Lipschitz on every bounded interval $[a,T] \subset [a,\infty)$: For all $a \le t_1 \le t_2 \le T$

$$||\nu_{t_1}|\,(E) - |\nu_{t_2}|\,(E)| \le |\nu_{t_1} - \nu_{t_2}|(E) \le \int_{t_1}^{t_2} \|\mu_s\|\,\mathrm{d}s \le C_T |t_1 - t_2|$$

and so $t \mapsto |\nu_t|(E)$ is differentiable for almost every $t \in [a,\infty)$ and satisfies

$$|\nu_t|(E) = |\nu_a|(E) + \int_a^t \frac{\mathrm{d}}{\mathrm{d}s}|\nu_s|(E)\mathrm{d}s \quad \forall\, t \in [a,\infty)\,.$$

Therefore in order to prove (5.5) we only have to show that for every Borel set $E \subset \mathbb{R}^N$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nu_t|(E) = \int_E h_t \mathrm{d}\mu_t\,, \quad \text{a.e.} \quad t \in [a,\infty) \tag{5.6}$$

which also implies that $t \mapsto \int_E h_t \mathrm{d}\mu_t$ belongs to $L^1_{\rm loc}([a,\infty))$.

For any $t, s \in [a,\infty)$, using

$$|\nu_s|(E) = \int_E \mathrm{d}|\nu_s| \ge \int_E h_t \mathrm{d}\nu_s$$

we have

$$|\nu_s|(E) - |\nu_t|(E) \ge \int_E h_t \mathrm{d}(\nu_s - \nu_t)\,. \tag{5.7}$$

Now take any $t\in(a,\infty)$ such that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}|\nu_t|(E)$ exists. By (5.7) we have

$$s>t \Longrightarrow \frac{|\nu_s|(E)-|\nu_t|(E)}{s-t}\ge \int_E h_t\mathrm{d}\Big(\frac{\nu_s-\nu_t}{s-t}\Big),$$
$$s<t \Longrightarrow \frac{|\nu_s|(E)-|\nu_t|(E)}{s-t}\le \int_E h_t\mathrm{d}\Big(\frac{\nu_s-\nu_t}{s-t}\Big).$$

Since $(\nu_s-\nu_t)/(s-t)\to\mu_t\ (s\to t)$ in norm $\|\cdot\|$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nu_t|(E)=\lim_{s\to t}\frac{|\nu_s|(E)-|\nu_t|(E)}{s-t}=\int_E h_t\mathrm{d}\mu_t\,.$$

This proves (5.6) and completes the proof. □

Let us now prove that the sign decomposition on differences of product measures preserves the invariance by exchanging v and v_*.

Lemma 5.2 (Sign decomposition and exchange of particles)**.** *For any $\mu,\nu\in\mathcal{B}^+_s(\mathbb{R}^N)$ ($s\ge 0$) and any locally bounded Borel function $\psi\in L^\infty_{-s}(\mathbb{R}^N\times\mathbb{R}^N)$ we have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v,v_*)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v_*,v)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)\,,\tag{5.8}$$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v,v_*)\mathrm{d}|\mu\otimes\mu-\nu\otimes\nu|=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v_*,v)\mathrm{d}|\mu\otimes\mu-\nu\otimes\nu|\,,\tag{5.9}$$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v,v_*)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)^+=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v_*,v)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)^+\,.\tag{5.10}$$

Proof of Lemma 5.2. Equality (5.8) easily follows from Fubini's theorem. Equality (5.10) follows from (5.9) and the relation

$$\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)^+=\frac{1}{2}\Big(\mathrm{d}|\mu\otimes\mu-\nu\otimes\nu|+\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)\Big)\,.$$

So we only have to prove (5.9). To do this we split ψ as $\psi=\psi^+-(-\psi)^+$ so that we can assume that $\psi\ge 0$. Let $h(v,v_*)$ be the sign function of the measure $\mu\otimes\mu-\nu\otimes\nu$. Then applying (5.8) to $\psi(v,v_*)h(v,v_*)$ we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v,v_*)\mathrm{d}|\mu\otimes\mu-\nu\otimes\nu|=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v,v_*)h(v,v_*)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)$$
$$=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v_*,v)h(v_*,v)\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)\le\iint_{\mathbb{R}^N\times\mathbb{R}^N}\psi(v_*,v)\mathrm{d}|\mu\otimes\mu-\nu\otimes\nu|\,.$$

Replacing $\psi(v,v_*)$ with $\psi(v_*,v)$ we also obtain the reversed inequality. This proves (5.9). □

Finally let us prove a signed estimate on the collision operator.

Lemma 5.3. *Let $B(z,\sigma)$ be given by (1.4)-(1.5)-(1.6) with $b(\cdot)$ satisfying* **(H4)***. Let $\mu\in\mathcal{B}^+_{2+\gamma}(\mathbb{R}^N)\,,\nu\in\mathcal{B}^+_{2\gamma}(\mathbb{R}^N)$, and let $h(v)$ be the sign function of $\mu-\nu$ and let $\kappa=\frac{1}{2}(1+h)$*

so that $\kappa \mathrm{d}(\mu-\nu) = \mathrm{d}(\mu-\nu)^+$. *Then for any* $\varphi \in C_b(\mathbb{R}^N)$ *satisfying* $0 \le \varphi(v) \le \langle v\rangle^2$ *we have*

$$\begin{aligned}(5.11)\qquad & \int_{\mathbb{R}^N} \varphi(v)\kappa(v)\mathrm{d}(Q(\mu,\mu) - Q(\nu,\nu))(v) \\ & \le E_\varphi + 2^{\gamma/2} A_0 \Big(\|\mu\|_{2+\gamma} \|\mu-\nu\|_0 + \|\mu\|_2 \|\mu-\nu\|_\gamma \Big)\end{aligned}$$

where

$$E_\varphi = A_0 2^\gamma \|\mu\|_\gamma \int_{\mathbb{R}^N} (\langle v\rangle^2 - \varphi(v))\langle v\rangle^\gamma \mathrm{d}\mu(v).$$

Proof of Lemma 5.3. Since φ is bounded, there is no problem of integrability in the following derivation. For instance we can write

$$(5.12)\qquad \int_{\mathbb{R}^N} \varphi(v)\kappa(v)\mathrm{d}\Big(Q(\mu,\mu) - Q(\nu,\nu)\Big)(v) = I^{(+)} - I^{(-)}$$

where

$$\begin{aligned} I^{(+)} &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} L_B[\varphi\kappa](v,v_*)\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu), \\ I^{(-)} &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\kappa(v)\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu).\end{aligned}$$

By definition of $B(v-v_*,\sigma)$ and $\varphi(v)\kappa(v) \le \langle v\rangle^2$ we have

$$\begin{aligned} & L_B[\varphi\kappa](v,v_*) + L_B[\varphi\kappa](v_*,v) \\ & \le \int_{\mathbb{S}^{N-1}} B(v-v_*,\sigma)(\langle v'\rangle^2 + \langle v'_*\rangle^2) d\sigma = A(v-v_*)(\langle v\rangle^2 + \langle v_*\rangle^2).\end{aligned}$$

Then using $\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu) \le \mathrm{d}(\mu\otimes\mu - \nu\otimes\nu)^+$ and Lemma 5.2 we compute

$$\begin{aligned} I^{(+)} \ &\le\ \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} (L_B[\varphi\kappa](v,v_*) + L_B[\varphi\kappa](v_*,v))\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu)^+ \\ &\le\ \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\langle v\rangle^2 \mathrm{d}(\mu\otimes\mu - \nu\otimes\nu)^+ .\end{aligned}$$

Since $A(v-v_*) \le A_0 2^\gamma \langle v\rangle^\gamma \langle v_*\rangle^\gamma$, $\langle v\rangle^2 - \varphi(v) \ge 0$, and $(\mu\otimes\mu - \nu\otimes\nu)^+ \le \mu\otimes\mu$, it follows that

$$\begin{aligned} & \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)(\langle v\rangle^2 - \varphi(v))\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu)^+ \\ & \le A_0 2^\gamma \iint_{\mathbb{R}^N\times\mathbb{R}^N} \langle v\rangle^\gamma \langle v_*\rangle^\gamma (\langle v\rangle^2 - \varphi(v))\mathrm{d}(\mu\otimes\mu) \\ & = A_0 2^\gamma \|\mu\|_\gamma \int_{\mathbb{R}^N} \langle v\rangle^\gamma (\langle v\rangle^2 - \varphi(v))\mathrm{d}\mu(v) = E_\varphi .\end{aligned}$$

Therefore using

$$\mathrm{d}(\mu\otimes\mu - \nu\otimes\nu)^+(v,v_*) \le \mathrm{d}\mu(v)\mathrm{d}(\mu-\nu)^+(v_*) + \mathrm{d}(\mu-\nu)^+(v)\mathrm{d}\nu(v_*)$$

we have

$$I^{(+)} \leq E_\varphi + \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\mathrm{d}\mu(v)\mathrm{d}(\mu-\nu)^+(v_*) + \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\mathrm{d}(\mu-\nu)^+(v)\mathrm{d}\nu(v_*)\,. \tag{5.13}$$

Similarly using $\mathrm{d}(\mu\otimes\mu-\nu\otimes\nu)(v,v_*) = \mathrm{d}\mu(v)\mathrm{d}(\mu-\nu)(v_*) + \mathrm{d}(\mu-\nu)(v)\mathrm{d}\nu(v_*)$ and $\kappa(v)\mathrm{d}(\mu-\nu)(v) = \mathrm{d}(\mu-\nu)^+(v)$ we have

$$I^{(-)} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\kappa(v)\mathrm{d}\mu(v)\mathrm{d}(\mu-\nu)(v_*) + \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\mathrm{d}(\mu-\nu)^+(v)\mathrm{d}\nu(v_*). \tag{5.14}$$

Canceling the common term in (5.13) and (5.14) and noticing that

$$\mathrm{d}(\mu-\nu)^+(v_*) \leq \mathrm{d}(\mu-\nu)(v_*) + \mathrm{d}|\mu-\nu|(v_*)$$

we obtain from (5.12),(5.13),(5.14) that

$$\int_{\mathbb{R}^N}\varphi(v)\kappa(v)\mathrm{d}(Q(\mu,\mu)-Q(\nu,\nu)) \leq E_\varphi + \iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\mathrm{d}\mu(v)\mathrm{d}|\mu-\nu|(v_*)\,. \tag{5.15}$$

Since $A(v-v_*)\varphi(v) \leq A_0 2^{\gamma/2}(\langle v\rangle^\gamma + \langle v_*\rangle^\gamma)\langle v\rangle^2$, it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} A(v-v_*)\varphi(v)\mathrm{d}\mu(v)\mathrm{d}|\mu-\nu|(v_*) \leq A_0 2^{\gamma/2}(\|\mu\|_{2+\gamma}\|\mu-\nu\|_0 + \|\mu\|_2\|\mu-\nu\|_\gamma)$$

which together with (5.15) proves (5.11). □

5.2. **Proof of Theorem 1.5.** We shall consider each part step by step.

Proof of part (a). Recall that $B(z,\sigma) = |z|^\gamma b(\cos\theta)$ satisfies $A_0 < \infty$ and $0 < \gamma \leq 2$. Let F_t be a conservative measure weak solution of Eq. (1.1) with $F_t|_{t=0} = F_0 \in \mathcal{B}_2^+(\mathbb{R}^N)$. We prove that F_t is a measure strong solution.

First of all by $\|F_t\|_0,\ \|F_t\|_\gamma \leq \|F_0\|_2$ and Proposition 2.3 we have

$$\|Q^\pm(F_t,F_t)\|_0 \leq 4A_0\|F_0\|_2^2\,, \quad \forall\, t\geq 0,$$

$$\langle Q(F_t,F_t),\,\varphi\rangle = \int_{\mathbb{R}^N}\varphi\mathrm{d}Q(F_t,F_t) \quad \forall\,\varphi\in C_b^2(\mathbb{R}^N), \quad \forall\, t\geq 0.$$

Since

$$t\mapsto \int_{\mathbb{R}^N}\varphi\mathrm{d}Q(F_t,F_t) = \langle Q(F_t,F_t),\,\varphi\rangle \ \text{ belongs to } \ C((0,\infty))\cap L^1_{\mathrm{loc}}([0,\infty))$$

there is no problem of integrability and the integral equation for a measure weak solutions becomes

$$\int_{\mathbb{R}^N}\varphi\mathrm{d}F_t = \int_{\mathbb{R}^N}\varphi\mathrm{d}F_0 + \int_0^t\mathrm{d}s\int_{\mathbb{R}^N}\varphi\mathrm{d}Q(F_s,F_s)\,. \tag{5.16}$$

Now take any $\varphi \in C_c^2(\mathbb{R}^N)$ satisfying $\|\varphi\|_{L^\infty} \le 1$. We have

$$\Big|\int_{\mathbb{R}^N}\varphi \mathrm{d}Q(F_t,F_t)\Big| \le \|Q(F_t,F_t)\|_0 \le 8A_0\|F_0\|_2^2\,, \quad \forall\, t\ge 0\,.$$

and thus using (5.16), for all $0 \le t_1 < t_2 < \infty$

$$\left|\int_{\mathbb{R}^N}\varphi \mathrm{d}(F_{t_2}-F_{t_1})\right| \le \int_{t_1}^{t_2}\Big|\int_{\mathbb{R}^N}\varphi \mathrm{d}Q(F_s,F_s)\Big| ds \le 8A_0\|F_0\|_2^2|t_1-t_2|\,.$$

Applying (1.19) this gives

$$\|F_{t_1}-F_{t_2}\|_0 \le 8A_0\|F_0\|_2^2|t_1-t_2|\,, \quad \forall\, t_1,t_2\in[0,\infty) \tag{5.17}$$

which enables us to prove the strong continuity:

$$t\mapsto F_t \in C([0,\infty);\mathcal{B}_2(\mathbb{R}^N)), \quad t\mapsto Q^{\pm}(F_t,F_t)\in C([0,\infty);\mathcal{B}_0(\mathbb{R}^N))\,. \tag{5.18}$$

In fact applying the inequality (2.19) in Proposition 2.3 with $s=0$ (recall that $0<\gamma\le 2$) we have

$$\|Q^{\pm}(F_t,F_t)-Q^{\pm}(F_{t_0},F_{t_0})\|_0 \le 8A_0\|F_0\|_2\|F_t-F_{t_0}\|_2, \quad t,t_0\ge 0. \tag{5.19}$$

Fix $t_0\in[0,\infty)$. Using (5.1), the conservation of mass and energy, $\mathrm{d}(F_{t_0}-F_t)^+ \le \mathrm{d}F_{t_0}$, and (5.17) we have for any $R\ge 1$

$$\begin{aligned}\|F_t-F_{t_0}\|_2 &= 2\int_{\mathbb{R}^N}\langle v\rangle^2\mathrm{d}(F_{t_0}-F_t)^+(v)\\ &\le 2R^2\int_{\langle v\rangle\le R}\mathrm{d}(F_{t_0}-F_t)^+(v) + 2\int_{\langle v\rangle>R}\langle v\rangle^2\mathrm{d}F_{t_0}(v)\\ &\le 2^4A_0R^2|t-t_0| + 2\int_{\langle v\rangle>R}\langle v\rangle^2\mathrm{d}F_{t_0}(v)\,.\end{aligned}$$

Thus first letting $t\to t_0$ and then letting $R\to\infty$ leads to $\limsup_{t\to t_0}\|F_t-F_{t_0}\|_2 = 0$. This together with (5.19) proves (5.18).

From the strong continuity in (5.18) we have for all $\varphi\in C_b^2(\mathbb{R}^N)$

$$\int_0^t \mathrm{d}s\int_{\mathbb{R}^N}\varphi \mathrm{d}Q(F_s,F_s) = \int_{\mathbb{R}^N}\varphi \mathrm{d}\left(\int_0^t Q(F_s,F_s)ds\right)$$

which together with (5.16) yields

$$\int_{\mathbb{R}^N}\varphi \mathrm{d}F_t = \int_{\mathbb{R}^N}\varphi \mathrm{d}F_0 + \int_{\mathbb{R}^N}\varphi \mathrm{d}\left(\int_0^t Q(F_s,F_s)ds\right)\,.$$

Therefore applying (1.19) we obtain

$$F_t = F_0 + \int_0^t Q(F_s,F_s)\mathrm{d}s\,, \quad t\ge 0\,.$$

Since $t\mapsto Q^{\pm}(F_t,F_t)\in C([0,\infty);\mathcal{B}_0(\mathbb{R}^N))$, it follows that $t\mapsto F_t\in C^1([0,\infty);\mathcal{B}_0(\mathbb{R}^N))$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}F_t = Q(F_t,F_t), \quad t\ge 0\,.$$

So F_t is a measure strong solution.

The converse is obvious because of (1.21) and (2.18) with $s=0$: Every measure strong solution is a measure weak solution.

Proof of parts (b)-(c)-(d). The proof of these three parts can be reduced to the proof of the following lemma:

Lemma 5.4. *Let $F_0 \in \mathcal{B}_2^+(\mathbb{R}^N)$ with $\|F_0\|_0 \neq 0$ and let F_t be a conservative measure strong solution of Eq.* (1.1) *with the initial datum F_0 and satisfy the moment production estimate* (1.24)-(1.25) *in Theorem 1.3. Let G_t be any measure strong solution of Eq.* (1.1) *on the time interval $[\tau, \infty)$ with initial data*

$$G_t|_{t=\tau} = G_\tau \in \mathcal{B}_2^+(\mathbb{R}^N)$$

for some $\tau \geq 0$, and satisfying $\|G_t\|_2 \leq \|G_\tau\|_2$ for all $t \in [\tau, \infty)$.

Then the stability estimates (1.27) *(for $\tau = 0$) and* (1.28) *(for $\tau > 0$) hold true.*

Note that the existence of such a solution F_t as in the statement has been proven by Theorem 1.3 and part (a) of the present theorem. Therefore if Lemma 5.4 holds true, then by taking $G_0 = F_0$ (for the case $\tau = 0$) we get $G_t \equiv F_t$ on $[0, \infty)$ and hence this proves parts (b), (c) and (d).

Proof of Lemma 5.4. Our proof is divided into several steps. First of all for notation convenience we denote

$$H_t = F_t - G_t.$$

Step 1. Given any $0 < r \in [\tau, \infty)$. We prove that

$$\begin{aligned} \|H_t\|_2 \leq\ & \|G_\tau\|_2 - \|F_\tau\|_2 + 2\|(H_r)^+\|_2 \\ & + 4A_0 \left(\mathcal{K}_{2+\gamma}(F_0) \int_r^t (1 + 1/s)\|H_s\|_0 \mathrm{d}s + \|F_0\|_2 \int_r^t \|H_s\|_\gamma \mathrm{d}s \right), \quad t \geq r\,. \end{aligned} \tag{5.20}$$

Here $\mathcal{K}_{2+\gamma}(F_0)$ is the constant in (1.25) with $s = 2 + \gamma$. To prove (5.20), we consider approximation: By $\mathrm{d}|H_t| = \mathrm{d}G_t - \mathrm{d}F_t + 2\mathrm{d}(H_t)^+$ we have

$$\|H_t\|_2 = \|G_t\|_2 - \|H_t\|_2 + 2 \lim_{n\to\infty} \int_{\mathbb{R}^N} \langle v\rangle_n^2 \mathrm{d}(H_t)^+ \quad \text{with} \quad \langle v\rangle_n^2 = \min\{\langle v\rangle^2, n\}.$$

Let $v \mapsto h_t(v)$ be the sign function of H_t and $\kappa_t(v) = \frac{1}{2}(1 + h_t(v))$ so that $\kappa_t \mathrm{d}H_t = \mathrm{d}(H_t)^+$. Then applying Lemma 5.1 to the measure

$$H_t = H_r + \int_r^t (Q(F_s, F_s) - Q(G_s, G_s))\mathrm{d}s \quad \text{for} \quad t \geq r$$

and then using Lemma 5.3 we have

$$\begin{aligned} \int_{\mathbb{R}^N} \langle v\rangle_n^2 \mathrm{d}(H_t)^+ &= \int_{\mathbb{R}^N} \langle v\rangle_n^2 \mathrm{d}(H_r)^+ + \int_r^t \mathrm{d}s \int_{\mathbb{R}^N} \langle v\rangle_n^2 \kappa_s(v) \mathrm{d}\Big(Q(F_s, F_s) - Q(G_s, G_s)\Big) \\ &\leq \|(H_r)^+\|_2 + E_n(t) + 2A_0 \Big(\int_r^t \|F_s\|_{2+\gamma} \|H_s\|_0 \mathrm{d}s + \|F_0\|_2 \int_r^t \|H_s\|_\gamma \mathrm{d}s \Big), \quad t \in [r, \infty) \end{aligned}$$

where

$$E_n(t) = 4A_0 \int_r^t \|F_s\|_\gamma \left(\int_{\mathbb{R}^N} (\langle v\rangle^2 - \langle v\rangle_n^2) \langle v\rangle^\gamma \mathrm{d}F_s \right) \mathrm{d}s.$$

Since, by moment estimate (1.24),

$$\int_r^t \|F_s\|_\gamma \left(\int_{\mathbb{R}^N} \langle v\rangle^{2+\gamma} \mathrm{d}F_s(v) \right) \mathrm{d}s \leq \|F_0\|_2 \int_r^t \|F_s\|_{2+\gamma} \mathrm{d}s < \infty\,, \quad t \in [r, \infty)\,,$$

it follows from dominated convergence that $\lim_{n\to\infty} E_n(t) = 0$ and thus

$$\begin{aligned}\|H_t\|_2 &\le \|G_t\|_2 - \|F_t\|_2 + 2\|(H_r)^+\|_2 \\ &+ 4A_0\Big(\int_r^t \|F_s\|_{2+\gamma}\|H_s\|_0 \mathrm{d}s + \|F_0\|_2 \int_r^t \|H_s\|_\gamma \mathrm{d}s\Big), \quad \forall\, t \in [r,\infty).\end{aligned}$$

By assumption on F_t and G_t we have $\|G_t\|_2 - \|F_t\|_2 \le \|G_\tau\|_2 - \|F_\tau\|_2$ and $\|F_s\|_{2+\gamma} \le \mathcal{K}_{2+\gamma}(F_0)(1 + 1/s)$. This proves (5.20).

Step 2. Suppose $\tau > 0$. Then taking $r = \tau$ in (5.20) and using $\|G_\tau\|_2 - \|F_\tau\|_2 + 2\|(H_\tau)^+\|_2 = \|H_\tau\|_2$ we obtain

$$\|H_t\|_2 \le \|H_\tau\|_2 + c_\tau \int_\tau^t \|H_s\|_2 \mathrm{d}s \qquad \forall\, t \in [\tau,\infty)$$

with $c_\tau = 4A_0(\mathcal{K}_{2+\gamma}(F_0) + \|F_0\|_2)(1 + \frac{1}{\tau})$. This gives (1.28) by Gronwall's Lemma.

The remaining steps deal with the case $\tau = 0$ and prove (1.27).

Step 3. If $\|H_0\|_2 \ge 1$, then using $\|F_t\|_2 = \|F_0\|_2$, $\|G_t\|_2 \le \|G_0\|_2$ we have

$$\|H_t\|_2 \le (1 + 2\|F_0\|_2)\|H_0\|_2 \qquad \forall\, t \in [0,\infty).$$

So in the following we assume that $\|H_0\|_2 < 1$. Note that in this case we have

$$\|F_t \pm G_t\|_2 \le 1 + 2\|F_0\|_2 =: C_0 \qquad \forall\, t \ge 0. \tag{5.21}$$

Using Proposition 2.3 we have

$$\begin{aligned}\|H_t\|_0 &\le \|H_0\|_0 + \int_0^t \|Q(F_s,F_s) - Q(G_s,G_s)\|_0 \mathrm{d}s \\ &\le \|H_0\|_0 + 4A_0 \int_0^t \Big(\|F_s + G_s\|_\gamma \|H_s\|_0 + \|F_s + G_s\|_0 \|H_s\|_\gamma\Big)\mathrm{d}s\end{aligned}$$

and thus by $0 < \gamma \le 2$ and (5.21) we obtain

$$\|H_t\|_0 \le \|H_0\|_0 + 8A_0C_0 \int_0^t \|H_s\|_2 \mathrm{d}s, \quad \forall\, t \ge 0. \tag{5.22}$$

Step 4. Let $r > 0$ satisfy $\|H_0\|_2 \le r \le 1$. We prove that

$$U(r) := \sup_{0\le t\le r} \|H_t\|_2 \le 4(1 + 9A_0C_0^2)\Psi_{F_0}(r). \tag{5.23}$$

First of all using (5.1) and $\|G_t\|_2 - \|F_t\|_2 \le \|G_0\|_2 - \|F_0\|_2 \le r$ we have

$$\|H_t\|_2 = \|G_t\|_2 - \|F_t\|_2 + 2\|(H_t)^+\|_2 \le r + 2\|(H_t)^+\|_2 \tag{5.24}$$

and for any $R \ge 1$

$$2\|(H_t)^+\|_2 \le 4R^2\|H_t\|_0 + 2\int_{|v|>R} \langle v\rangle^2 \mathrm{d}F_t(v). \tag{5.25}$$

Next by $\|H_0\|_2 \le r$ and (5.22) we have

$$4R^2\|H_t\|_0 \le 4(1 + 8A_0C_0^2)R^2\, r \quad \forall\, t \in [0,r]. \tag{5.26}$$

Using the conservation of mass and energy we compute

$$\begin{aligned}\int_{|v|>R}\langle v\rangle^2 \mathrm{d}F_t(v) &= \int_{\mathbb{R}^N}\langle v\rangle^2 \mathrm{d}F_t(v)-\int_{|v|\le R}\langle v\rangle^2 \mathrm{d}F_t(v)\\ &= \int_{\mathbb{R}^N}\langle v\rangle^2 \mathrm{d}F_0(v)-\int_{|v|\le R}\langle v\rangle^2 \mathrm{d}F_0(v)-\int_0^t \mathrm{d}s\int_{|v|\le R}\langle v\rangle^2 \mathrm{d}Q(F_s,F_s)\\ &\le \int_{|v|>R}\langle v\rangle^2 \mathrm{d}F_0(v)+\int_0^t \mathrm{d}s\int_{|v|\le R}\langle v\rangle^2 \mathrm{d}Q^-(F_s,F_s)\,.\end{aligned}$$

For the last term we use $|v-v_*|^\gamma\le \langle v\rangle^\gamma\langle v_*\rangle^\gamma\le\langle v\rangle^2\langle v_*\rangle^2$ to get for all $t\in[0,r]$

$$\int_0^t \mathrm{d}s\int_{|v|\le R}\langle v\rangle^2 \mathrm{d}Q^-(F_s,F_s)\le 2R^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^N}\mathrm{d}Q^-(F_s,F_s)\le 2A_0\|F_0\|_2^2R^2\,r\,.$$

Thus

$$\int_{|v|>R}\langle v\rangle^2 \mathrm{d}F_t(v)\le \int_{|v|>R}\langle v\rangle^2 \mathrm{d}F_0(v)+2A_0\,\|F_0\|_2^2R^2\,r\quad \forall\, t\in[0,r]\,. \tag{5.27}$$

Combining (5.25)-(5.26)-(5.27) gives

$$2\|(H_t)^+\|_2\le 4(1+9A_0C_0^2)R^2r+4\int_{|v|>R}|v|^2\mathrm{d}F_0(v)\,,\quad t\in[0,r]\,. \tag{5.28}$$

Now choose $R=r^{-1/3}$. Then from (5.24), (5.28) we obtain

$$\|H_t\|_2\le r+4(1+9A_0C_0^2)r^{1/3}+4\int_{|v|>r^{-1/3}}|v|^2\mathrm{d}F_0(v)\,,\quad t\in[0,r]\,.$$

This gives (5.23) by definition of $\Psi_{F_0}(r)$ in (1.22).

Step 5. In the following we denote $C_i=\mathcal{R}_i(\gamma,A_0,A_2,\|F_0\|_0,\|F_0\|_2)$ for $(i=1,2,\dots,6)$, where $\mathcal{R}_i(x_1,x_2,\dots,x_5)$ are some explicit positive continuous functions in $(\mathbb{R}_{>0})^5$.

In (5.20) setting $\tau=0, r=1$ we have

$$\|H_t\|_2\le\|H_0\|_2+2\|H_1\|_2+C_1\int_1^t\|H_s\|_2\mathrm{d}s\,,\quad t\ge 1$$

so that Gronwall's Lemma applies to get

$$\|H_t\|_2\le\Big(\|H_0\|_2+2\|H_1\|_2\Big)\exp(C_1(t-1))\,,\quad t\ge 1\,. \tag{5.29}$$

Now we concentrate our estimate for $t\in[0,1]$. In what follows we assume r satisfy

$$r>0\,,\quad \|H_0\|_2\le r<1\,. \tag{5.30}$$

Using (5.20) (with $\tau=0$), $\|G_0\|_2-\|F_0\|_2\le\|H_0\|_2\le r$, and $\|H_r\|_2\le U(r)$ we have

$$\|H_t\|_2\le r+2U(r)+C_2\left(\int_r^t\frac{1}{s}\|H_s\|_0\mathrm{d}s+\int_r^t\|H_s\|_\gamma\mathrm{d}s\right),\quad t\in[r,1]\,.$$

Further, using (5.22) we compute for all $t\in[r,1]$

$$\begin{aligned}\int_r^t\frac{1}{s}\|H_s\|_0\mathrm{d}s &\le r\log(t/r)+8A_0C_0\int_r^t\frac{1}{s}\int_0^s\|H_\tau\|_2\mathrm{d}\tau\mathrm{d}s\\ &\le r|\log r|+8A_0C_0\int_0^t\|H_\tau\|_2|\log\tau|\mathrm{d}\tau\,.\end{aligned}$$

Thus for all $t \in [r, 1]$

$$\|H_t\|_2 \le r + 2U(r) + C_2 r|\log r| + C_3 \int_0^t \|H_s\|_2(1 + |\log s|)\mathrm{d}s\,. \tag{5.31}$$

Since $\|H_t\|_2 \le U(r)$ for all $t \in [0, r]$, the inequality (5.31) holds for all $t \in [0, 1]$. Therefore by Gronwall's Lemma we conclude

$$\|H_t\|_2 \le C_4(r + U(r) + r|\log r|\,) \qquad \forall\, t \in [0,\, 1]\,. \tag{5.32}$$

In particular taking $t = 1$ yields the estimate for $\|H_1\|_2$ and thus from (5.29)-(5.30) we obtain

$$\|H_t\|_2 \le C_5(r + U(r) + r|\log r|\,)\exp(C_1(t-1))\,, \quad \forall\, t \in [1, \infty)\,. \tag{5.33}$$

Combining (5.32)-(5.33) and the inequality $r|\log r| \le r^{1/3}$ we conclude

$$\|H_t\|_2 \le \Psi_{F_0}(r)\exp(C_6(1+t)) \quad \forall\, t \ge 0\,. \tag{5.34}$$

Finally if $\|H_0\|_2 = 0$, then in (5.34) letting $r \to 0+$ leads to $\|H_t\|_2 \equiv 0$; if $\|H_0\|_2 > 0$, we take $r = \|H_0\|_2$. This proves (1.27) and completes the proof of the lemma. □

Proof of part (e). Let $\mathrm{d}F_0(v) = f_0(v)\mathrm{d}v$ with $0 \le f_0 \in L^1_2(\mathbb{R}^N)$, and let F_t be the unique conservative measure strong solution of Eq. (1.1) with the initial datum F_0. By the Lebesgue-Radon-Nikodym theorem, for every $t \ge 0$ we have a decomposition $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v + \mathrm{d}\mu_t(v)$ where $0 \le f_t \in L^1_2(\mathbb{R}^N)$, $\mu_t \in \mathcal{B}^+_2(\mathbb{R}^N)$ and μ_t concentrates on a Lebesgue null set. By the uniqueness of F_t we can assume that $\|f_0\|_{L^1} \ne 0$. Let

$$f^n_0(v) = \min\{f_0(v),\, n\}e^{-|v|^2/n}\,, \quad \text{and } \mathrm{d}F^n_0(v) = f^n_0(v)\mathrm{d}v\,.$$

By Step 2 of the proof of Theorem 1.3, for every n there is a conservative measure weak solution F^n_t with the initial datum F^n_0 and $\mathrm{d}F^n_t(v) = f^n_t(v)\mathrm{d}v$, $0 \le f^n_t \in L^1_2(\mathbb{R}^N)$ for all $t \ge 0$. By part (a), F^n_t is also a measure strong solution. Since $\mathrm{d}(F_t - F^n_t) = (f_t - f^n_t)\mathrm{d}v + \mathrm{d}\mu_t$ we have $\|F_t - F^n_t\|_2 = \|f_t - f^n_t\|_{L^1_2} + \|\mu_t\|_2$. Since

$$\begin{aligned} \|F_0 - F^n_0\|_2 &= \|f_0 - f^n_0\|_{L^1_2} \\ &\le \int_{f_0(v)>n} f_0(v)\langle v\rangle^2\mathrm{d}v + \int_{\mathbb{R}^N} f_0(v)(1 - e^{-|v|^2/n})\langle v\rangle^2\mathrm{d}v \to 0 \quad (n \to \infty) \end{aligned}$$

it follows from the stability estimate that for every fixed $t \ge 0$ we have

$$\|f_t - f^n_t\|_{L^1_2} + \|\mu_t\|_2 = \|F_t - F^n_t\|_2 \le e^{C(1+t)}\Psi_{F_0}(\|F_0 - F^n_0\|_2) \xrightarrow[n\to 0]{} 0$$

and therefore $\mu_t \equiv 0$. Thus $\mathrm{d}F_t(v) = f_t(v)\mathrm{d}v$ for all $t \ge 0$ and hence f_t is the unique conservative mild solution of Eq. (1.1) associated with the initial datum f_0. This proves part (e).

Proof of part (f). Suppose $F_0 \in \mathcal{B}^+(\mathbb{R}^N)$ is not a Dirac mass. We can assume that $\|F_0\|_0 \neq 0$. Let $f_0^n(v)$ be defined by (4.21)-(4.23) (the Mehler transform of F_0). By part (e), for every $n \geq 1$ there exists a unique conservative L^1-solution f_t^n of Eq.(1.1) associated with the initial datum $f_t^n|_{t=0} = f_0^n$. If we define F_0^n, F_t^n by $\mathrm{d}F_0^n(v) = f_0^n(v)\mathrm{d}v$ and $\mathrm{d}F_t^n(v) = f_t^n(v)\mathrm{d}v$, then by uniqueness and Theorem 1.3 we see that F_t^n satisfies the moment production estimates. Thus it is easily checked that the Step 3 (where there is no need of introducing $\widetilde{f}_0^n$ for the present case) in the proof of Theorem 1.3 is totally valid here. Therefore there is a subsequence, which we still denote as $\{f_t^n\}_{n=1}^{\infty}$, such that for the unique measure solution F_t of Eq. (1.1) with $F_t|_{t=0} = F_0$, the weak convergence (1.30) holds true. This completes the proof of Theorem 1.5.

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